Take-Home Exam, Week 2

Honor Code Guidelines for Take-Home Exams (from the Course Information Sheet):

Take-home exams are individual work, not group work.

The take-home exams are open-book (meaning open Course Reader, handouts, textbooks, course lecture videos, and internet searches for conceptual information e.g. Wikipedia). Consultation of other humans in any form or medium (e.g., communicating with classmates, asking questions on forum websites such as StackOverflow) is prohibited. All work done with the assistance of any material in any way (other than provided CS103 course materials) must include citation (e.g., "Referred to Wikipedia page on DeMorgan's Law for Question 2."). Copying solutions is unacceptable, even with citation. If you chance encounter solutions to the problem, navigate away from that page before you feel tempted to copy. Also, please never forget that because of the revise & resubmit policy, there is no reason to violate your conscience to complete a take-home exam.

If you become aware of any honor code violations by any student in the class, your commitments under the Stanford Honor Code obligate you to inform course staff.

Grading for Take-Home Exams:

To pass the course, you are required to answer all take-home exam questions correctly for the entire quarter. Correct means not only the “right” answer, but also, particularly for proofs, an answer that is expressed with the high style standards of precision and clarity that we expect in this class (see the Proofwriting Checklist handout).

However, we hasten to reassure you that if you do not pass a take-home exam, you do not automatically fail the course! Our graders will give you feedback on how your work could be improved by Tuesday following the take-home exam deadline, and you will have until noon that Saturday to revise and resubmit your exam. If you have any uncertainty or confusion about what was wrong or how to properly revise your work, please request help in office hours. We may also schedule targeted small-group help sessions (think “Problem 3 Intervention”) or similar resources, depending on what kinds of patterns of errors we see from individuals and across the class as a whole. The goal is to provide the flexibility and coaching you need to cross the finish line, despite what may be unusual, difficult, and unpredictable circumstances this quarter. Though, as with any course, ultimately you do need to take the steps to cross the finish line.

<table>
<thead>
<tr>
<th>Question</th>
<th>Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) Indirect Proof</td>
<td>10</td>
</tr>
<tr>
<td>(2) Set Theory</td>
<td>10</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>20</strong></td>
</tr>
</tbody>
</table>
Problem One: Indirect Proof

(10 Points)

Contrapositive vibes only

Recall the definition of modular congruence from Problem Set 1:

We say that \( a \equiv_b b \) if there exists an integer \( q \) such that \( a = b + kq \).

Theorem: For all integers \( a, b, c, d, \) and \( k \), if \( a + c \equiv_b b + d \), then \( a \not\equiv_b b \) or \( c \not\equiv_b d \).

i. (2pts) Write the **negation** of the theorem. Fully “distribute” the negation through the statement, as we learned in class (i.e., don’t just write “It is not the case that...” at the beginning and call it a day :-)).

ii. (2pts) Write the **contrapositive** of the theorem.

iii. (6pts) Prove the theorem, using **proof by contrapositive**. There are other ways to approach it, but we would like to see this style. *(Tip: Be sure to review the Proofwriting Checklist. The modular congruence problems on Pset1, including proof critique, should also be helpful references.)*
Problem Two: Set Theory

Splitting hairs

First we define a new operator \( \cap_M \) as follows:

For some set of sets \( T \) and some set \( S \), \( T \cap_M S = \{ t \cap S \mid t \in T \} \).

For example, if \( T_1 = \{ \emptyset, \{1\}, \{2\}, \{1, 2\}, \{1, 3\} \} \) and \( S_1 = \{1, 3, 4\} \), then \( T_1 \cap_M S_1 = \{ \emptyset, \{1\}, \{1, 3\} \} \), because:

\[
\emptyset \cap S_1 = \emptyset, \\
\{1\} \cap S_1 = \{1\}, \\
\{2\} \cap S_1 = \emptyset, \\
\{1, 2\} \cap S_1 = \{1\}, \text{ and} \\
\{1, 3\} \cap S_1 = \{1, 3\}.
\]

Now we define a new term split for some set of sets \( T \) and some set \( S \):

\( T \) splits \( S \) if \( \wp(S) \subseteq T \cap_M S \).

For example, for the value of \( T_1 \) given above and \( S_2 = \{1, 2\} \), \( T_1 \) splits \( S_2 \).

i. (2pts) Let \( T_3 = \{ \emptyset, \{0\}, \{1\}, \{0, 1\}, \{1, 2\}, \{0, 1, 2\} \} \). There are 6 distinct sets \( S \) such that \( T_3 \) splits \( S \). List them.

ii. (8pts) Prove this theorem using Direct Proof approach: For all sets \( B \) and \( C \), and all sets of sets \( T \), if \( B \subseteq C \) and \( T \) splits \( C \), then \( T \) splits \( B \). (Tip: Be sure to review the Guide to Set Theory Proofs and the Proofwriting Checklist. This proof is an excellent tour of about 4 different key proofwriting skills and techniques, but that does make it a bit challenging for Week 2 of the quarter. On the next page, we’ve provided a template to help steer you in the right direction. You are not required to use the template in that, if you want an extra challenge, you can try solving it without looking at the template. But you would of course still be required to follow the structure and style guidelines in our Guide to Set Theory Proofs and the Proofwriting Checklist.)
**Theorem:** For all sets $B$ and $C$, and all sets of sets $T$, if $B \subseteq C$ and $T$ splits $C$, then $T$ splits $B$.

**Proof:**

*If you use this template, remove all the blue text!*

This theorem is universally quantified, and in the form of an implication. So first we want to separate out our “assume” (where we will pick arbitrary value(s) for our universally quantified variables in the antecedent, subject to any criteria we assume hold from the antecedent) and our “want to show” (the consequent). What are they?

**Assume:** Pick arbitrary sets ________ and an arbitrary set of sets ______, such that _____________________.

*When stating the want to show, it often helps to immediately expand a definition in the want to show, as a way of being more specific about our destination, we’ll do that here.*

**Want to show:** We want to show that $T$ splits $B$ *(the consequent)*. That is, we want to show that __________ (expand using the definition of “splits”).

Since our want to show (expanded version) includes a statement that one set is a subset of another, we need to use our standard approach to proving subset. That is **element analysis**. We must pick an arbitrary element of the set on the left, and then our new subgoal “want to show” is to show that it is an element of the set on the right. **This is the only valid way to prove subsets in this class.**

Arbitrarily pick a set $A \in ______$ *(set on the left)*. We want to show that $A \in ______$ *(set on the right)*.

Now I’m going to skip about 3 sentences, which you can write yourself. You may find it helpful to draw a picture of what you know about $A$, $B$, and $C$ and this point. Hint: you may always cite facts that we proved in lecture, psets, or handouts; and in this case you’ll want to cite the Proofwriting Checklist handout where we proved that subset is transitive. By the end of these few sentences, you should arrive at a point where you have this:

Since $T$ splits $C$ and $A \in \wp(C)$, we know that $A \in T \cap M C$.

To continue, we will want to expand this fact using the definition of $\cap M$.

In other words, there exists a set $t \in _____$ such that ______________.

Now we need to look all the way back to what you wrote for the expanded “Want to show:” and actually expand it one step further. That gives us this new even more specified want to show:

We want to show that there exists a $t_1 \in _____$ such that ______________. We choose $t_1 = t$, meaning that we want to show that $t \cap B = A$. Yet another subgoal of our want to show! *(The constantly shifting want to show is the major reason we decided to give you a template. :-) )* A critical thing to remember here is our required procedure for proving set equality, which is to do it as two separate proofs (that left is subset of the right, and that right is subset of the left). And each of those proofs must use our subset element analysis described above! **This is the only valid way to prove set equality in this class.**

I personally found it very helpful to draw a picture of $A$, $B$, $C$, and $t$ at this point.

1. First we show that __________ : Continue proof here, being sure to use element analysis to show subset.

2. Next we show that __________ : Continue proof here, being sure to use element analysis to show subset.

You’ve now reached your stated want to show for the whole proof! So note that fact and conclude.