This exam is closed-book and closed-computer. You may have a double-sided, 8.5” × 11” sheet of notes with you when you take this exam. You may not have any other notes with you during the exam. You may not use any electronic devices (laptops, cell phones, etc.) during the course of this exam. Please write all of your solutions on this physical copy of the exam.

You are welcome to cite results from the problem sets or lectures on this exam. Just tell us what you're citing and where you're citing it from. However, please do not cite results that are beyond the scope of what we've covered in CS103.

SUNetID: ____________________________________________
Last Name: _________________________________________
First Name: _________________________________________

I accept both the letter and the spirit of the Honor Code. I have not received any unpermitted assistance on this test, nor will I give any. I do not have any advance knowledge of what questions will be asked on this exam. My answers are my own work. I understand that the Honor Code requires me to report any violations of the Honor Code that I witness during this exam.

(signed) ___________________________________________

You have three hours to complete this exam. There are 48 total points.

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Problem One: Mathematical Logic  

Relationship Advice, CS103 Edition

When we discussed first-order logic, we spent a decent amount of time talking about how to translate statements from English into first-order logic. You practiced this skill on Problem Set Two. In this problem, we'd like you to show us what you've learned about the art of first-order translation.

We’ve given a lot of dating advice thus far in CS103, and on this exam we thought that it might be fun to get you to think about interpersonal dynamics in the context of mathematical logic.

i. **(4 Points)** Given the predicates

   - Person(x), which states that x is a person;
   - Hobby(x), which states that x is a hobby; and
   - Enjoys(x, y), which states that x enjoys y,

translate the statement “for each person, there’s some other person who enjoys exactly the same set of hobbies” into first-order logic. (That might be a fun person to go on a date with!)
ii. **(4 Points)** Given the predicates

- \( \text{Person}(x) \), which states that \( x \) is a person;
- \( \text{Chore}(x) \), which states that \( x \) is a chore;
- \( \text{WillDate}(x, y) \), which states that \( x \) will date \( y \); and
- \( \text{WillTakeCareOf}(x, y) \), which states that \( x \) will take care of \( y \),

translate the statement “no one will date anyone who won’t take care of any chores” into first-order logic.
We've talked a lot about negating and simplifying statements in first-order logic, a useful skill with applications to disproofs, proofs by contradiction, and proofs by contrapositive. You practiced this on Problem Set Two, and in this question we'd like you to demonstrate what you've learned.

iii. **(4 Points)** Consider the following statement in first-order logic:

\[(\exists x. P(x)) \rightarrow (\forall y. (Q(y) \lor (\exists z. (R(y, z) \iff T))))\]

Give a statement in first-order logic that is the negation of this statement. As in Problem Set Two, your final formula must not have any negations in it, except for direct negations of predicates. Do not introduce any new predicates, functions, or constants.

While you're not required to show every step of your work, it does make it a bit easier for us to give you partial credit in the event that you accidentally make a mistake. If you do show each step, please clearly indicate what your final answer is.

As a courtesy to your TAs, please try to make the parenthesization of your final statement as clear as possible.
**Problem Two: Set Theory**

*Set Theory Sampler*

(12 Points)

The first part of this problem is a Set Theory Scavenger Hunt! For each of the following criteria, give an example of a set or group of sets matching those criteria. Express your sets using formal mathematical notation (e.g. $\emptyset$, or $\{ n \in \mathbb{N} \mid n < 137 \}$, or $\{ 1, 3, 7 \}$, but not “the set of all natural numbers less than 137” or $\{ 0, 1, 2, \ldots, 136 \}$)

i. **(2 Points)** Give a set $A$ and a set $B$ where both $A$ and $B$ are infinite sets (that is, they contain infinitely many elements) and $A \cap B$ is infinite. Briefly explain why $A \cap B$ is infinite.

ii. **(2 Points)** Give a set $A$ and a set $B$ where both $A$ and $B$ are infinite sets, but $A \cap B$ is finite (that is, $|A \cap B|$ is a natural number.) Briefly explain why $A \cap B$ is finite.

iii. **(2 Points)** Give four sets such that the intersection of any three of those sets is nonempty, but the intersection of all four of those sets is empty. Briefly justify your answer.

iv. **(3 Points)** Give a set $A$ where $A \cap \wp(A) \neq \emptyset$ and $A \cap \wp(A) \neq A$. Briefly justify your answer.
This last part of the problem has nothing to do with the first part. 😓

v. **(3 Points)** Let \( P \) be the set of all people who attend to their physical well-beings, \( E \) be the set of all people who attend to their emotional well-beings, and \( M \) be the set of people missing out on the full human experience. Write an expression using set theory notation (e.g. \( \in \), \( \subseteq \), \( \cap \), etc.) that says “everyone who attends to their physical well-being but not their emotional well-being (or vice-versa) is missing out on the full human experience.” Your answer should not involve set-builder notation, first-order logic, or plain English. For full credit, provide the simplest possible answer. Please do not introduce any new variables.
Problem Three: Proofwriting I

Proofs on Set Theory

(12 Points)

On Problem Sets One and Two, you played around with different properties of sets and wrote many proofs on set theory. This question is designed to let you show us what you’ve learned in the course of doing so.

Let $A$ and $B$ be arbitrary sets. Prove that $A \in \varnothing(B)$ if and only if $A \cap B = A$.

Some hints:

- The above statement contains a biconditional. How do you prove biconditional statements?
- Suppose you want to prove that $S = T$, where $S$ and $T$ are sets. What’s the easiest way to do this?

Feel free to use the space below for scratch work. There’s room to write your answer to this problem on the next page of this exam.
Problem Four: Proofwriting II  
(12 Points)

Constrained Tournament Winners

On Problem Set Two, you explored tournaments, contests between groups of \( n \) players. This problem explores another property of tournaments.

Let's begin by refreshing some definitions. A tournament is a contest among \( n \) players. Each player plays a game against each other player, and either wins or loses the game (let's assume that there are no draws). We can visually represent a tournament by drawing a circle for each player and drawing arrows between pairs of players to indicate who won each game. For example, in the tournament to the left, player \( A \) beat player \( E \), but lost to players \( B, C, \) and \( D \).

A tournament winner is a player in a tournament who, for each other player, either won her game against that player, or won a game against a player who in turn won his game against that player (or both). For example, in the graph on the left, players \( B, C, \) and \( E \) are tournament winners. However, player \( D \) is not a tournament winner, because he neither beat player \( C \), nor beat anyone who in turn beat player \( C \). Although player \( D \) won against player \( E \), who in turn won against player \( B \), who then won against player \( C \), under our definition player \( D \) is not a tournament winner. (Make sure you understand why!)

i. (4 Points) Let \( T \) be an arbitrary tournament. What is the contrapositive of the following statement? (You don’t need to expand out the definition of a tournament winner when writing out your answer.)

\[
\text{If } p \text{ and } q \text{ are tournament winners in } T, \\
\text{then at least one of } p \text{ and } q \text{ won more than one game in } T.
\]
As a refresher from the previous page, a **tournament winner** is a player in a tournament who, for each other player, either won her game against that player, or won a game against a player who in turn won his game against that player (or both).

ii. **(8 Points)** Let $T$ be an arbitrary tournament with *at least four players*, and let $p$ and $q$ be two different players in $T$. Prove the following statement using a **proof by contrapositive**:

If $p$ and $q$ are tournament winners in $T$,
then at least one of $p$ and $q$ won more than one game in $T$.

Some hints on this problem:

- You may want to draw some pictures to help build your intuition for what’s going on, though your final proof should be purely written and not involve any diagrams.
- Remember that $p$ and $q$ played a game against one another. You can assume, without loss of generality, that $p$ won that game.
- The fact that $T$ has at least four players in it is significant. This result isn’t true if the tournament has only three players.
- You may want to introduce some variables to refer to other players in the tournament.

Feel free to use the space below for scratch work. There’s room to write your answer to this problem on the next page of this exam.