Practice Second Midterm Exam 1

This exam is closed-book and closed-computer. You may have a double-sided, 8.5” × 11” sheet of notes with you when you take this exam. You may not have any other notes with you during the exam. You may not use any electronic devices (laptops, cell phones, etc.) during the course of this exam. Please write all of your solutions on this physical copy of the exam.

You are welcome to cite results from the problem sets or lectures on this exam. Just tell us what you're citing and where you're citing it from. However, please do not cite results that are beyond the scope of what we've covered in CS103.

On the actual exam, there'd be space here for you to write your name and sign a statement saying you abide by the Honor Code. We're not collecting or grading this exam (though you're welcome to step outside and chat with us about it when you're done!) and this exam doesn't provide any extra credit, so we've opted to skip that boilerplate.

You have three hours to complete this practice midterm. There are 48 total points. This practice midterm is purely optional and will not directly impact your grade in CS103, but we hope that you find it to be a useful way to prepare for the exam. You may find it useful to read through all the questions to get a sense of what this practice midterm contains before you begin.

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Problem One: Discrete Structures I  (12 Points)
(Midterm Exam, Winter 2016)

On Problem Set Three, you explored binary relations and formal definitions specified in first-order logic. This question explores a way of forming new binary relations and asks you to prove properties about them. In the course of doing so, we hope you'll be able to demonstrate what you’ve learned about binary relations and proving results about terms defined in first-order logic.

Let's suppose that we have an arbitrary strict order \( R \) over a set \( A \). We can use \( R \) to define a new binary relation \( \mathcal{R} \) over the set \( \mathcal{P}(A) \) as follows:

\[
\forall X \in \mathcal{P}(A), \forall Y \in \mathcal{P}(A). \quad X \mathcal{R} Y \iff Y \neq \emptyset \land \forall x \in X. \forall y \in Y. \ xRy.
\]

This relation \( \mathcal{R} \) is called the **lift of \( R \) to \( \mathcal{P}(A) \)**. As a reminder, the word “if” in the above definition means “is defined as” and is not an implication.

Prove that if \( R \) is an arbitrary strict order over a set \( A \), then \( \mathcal{R} \) is a strict order over the set \( \mathcal{P}(A) \).
(Extra space for your answer to Problem One, if you need it.)
Problem Two: Discrete Structures II

(Midterm Exam, Fall 2018)

(12 Points)

On Problem Set Four and Problem Set Five, you worked with graphs and played around with many of their properties. This question explores a new class of graphs and is designed to give you a chance to show us what you’ve learned along the way.

Let’s begin with a definition. An undirected graph \( G = (V, E) \) is called a **friendship graph** if it satisfies the following requirement:

For any nodes \( u, v \in V \) where \( u \neq v \), there is exactly one node \( z \in V \) where \( \{u, z\} \in E \) and \( \{v, z\} \in E \).

As a reminder, undirected graphs cannot have edges from nodes back to themselves.

i. **(4 Points)** Prove that if \( G = (V, E) \) is a friendship graph, then \( G \) does not contain any simple cycles of length four.

As a hint, draw pictures.
(Extra space for your answer to Problem Two, Part (i), if you need it.)
As a refresher from the previous page, an undirected graph $G = (V, E)$ is called a **friendship graph** if it satisfies the following requirement:

For any nodes $u, v \in V$ where $u \neq v$, there is **exactly one** node $z \in V$ where $\{u, z\} \in E$ and $\{v, z\} \in E$.

As a reminder, undirected graphs cannot have edges from nodes back to themselves.

ii. **(5 Points)** Let $G = (V, E)$ be a friendship graph with at least two nodes. Prove that for every node $v \in V$, there’s a simple cycle of length three that contains $v$.

As a hint, draw pictures.
(Extra space for your answer to Problem Two, Part (ii), if you need it.)
As a refresher, an undirected graph $G = (V, E)$ is called a **friendship graph** if it satisfies the following requirement:

For any nodes $u, v \in V$ where $u \neq v$, there is exactly one node $z \in V$ where $\{u, z\} \in E$ and $\{v, z\} \in E$.

As a reminder, undirected graphs cannot have edges from nodes back to themselves.

iii. (3 Points) In the space below, draw a friendship graph with exactly seven nodes. No justification is necessary. There's scratch space for this problem on the next page of this exam.

As a hint, use the results you proved in parts (i) and (ii) to guide your search.

We will grade whatever you draw in this box.
Feel free to use the space on the next page for scratch work.
(Scratch space for your answer to Problem Two, Part (iii), if you need it.)
Problem Three: Discrete Structures III  
(Midterm Exam, Fall 2018)

(12 Points)

On Problem Sets Three, Four, and Five, you explored properties of functions. On Problem Set Three, you worked with equivalence relations. This question explores the interplay of these concepts.

Let’s begin with a new definition. Given a set $A$, a **transformation group over $A$** is a set $\mathcal{F}$ where

- every element of $\mathcal{F}$ is a bijection whose domain and codomain are $A$;
- $\text{id}_A \in \mathcal{F}$, where $\text{id}_A : A \to A$ is the **identity bijection**, the bijection defined as $\text{id}_A(x) = x$;
- if $f \in \mathcal{F}$, then $f^{-1} \in \mathcal{F}$; and
- if $f, g \in \mathcal{F}$, then $g \circ f \in \mathcal{F}$.

If $\mathcal{F}$ is a transformation group over a set $A$, we can define a binary relation $\star_{\mathcal{F}}$ over $A$ as follows:

$$a \star_{\mathcal{F}} b \quad \text{if} \quad \exists f \in \mathcal{F}. f(a) = b.$$  

As a reminder, the “if” in the above statement means “is defined as” and is not an implication.

i. **(9 Points)** Let $\mathcal{F}$ be an arbitrary transformation group over some set $A$. Prove that $\star_{\mathcal{F}}$ is an equivalence relation over $A$.

Don’t let the notation scare you. Set this proof up as you would any other proof that a relation is an equivalence relation and you’ll discover how all these pieces fit together.
(Extra space for your answer to Problem Three, Part (i), if you need it.)
As a refresher, if $\mathcal{S}$ is a transformation group over a set $A$, we can define a binary relation $\star_{\mathcal{S}}$ over $A$ as follows:

$$a \star_{\mathcal{S}} b \text{ if } \exists f \in \mathcal{S}. f(a) = b.$$ 

On Problem Set Four and Problem Set Five, you explored automorphisms of graphs. It turns out that if $G = (V, E)$ is a graph, then the set of all automorphisms of $G$, denoted $\text{Aut}(G)$, is a transformation group. (You don’t need to prove this.)

ii. (3 Points) Below is a picture of a graph $G = (V, E)$, where $V = \{b, c, h, p, s, v\}$. What are the equivalence classes of the relation $\star_{\text{Aut}(G)}$ over the set $V$? Briefly justify your answer, but no formal proof is necessary.

As a hint, use your intuition behind what structure an automorphism captures.
Problem Four: Discrete Structures IV

(12 Points)

On Problem Set Five, you explored recurrence relations. This problem is designed to give you a chance to demonstrate what you’ve learned in the process.

First, two new definitions. First, if \( x \) and \( y \) are integers, we say \( x \) **divides** \( y \) if there is an integer \( q \) such that \( y = xq \). Second, we say that two integers \( a \) and \( b \) are **relatively prime** if the only integers that divide both \( a \) and \( b \) are \( \pm 1 \).

i. **(5 Points)** Let \( a \) and \( b \) be arbitrary integers. Prove that if \( a \) and \( b \) are relatively prime, then \( a+b \) and \( a \) are relatively prime.
(Extra space for your answer to Problem Four, Part (i), if you need it.)
The **Fibonacci numbers** are a sequence of natural numbers defined by the following recurrence relation:

\[
\begin{align*}
F_0 &= 0 \\
F_1 &= 1 \\
F_{n+2} &= F_{n+1} + F_n.
\end{align*}
\]

The first few terms of the Fibonacci sequence are

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, … .

This question explores a nifty property of the Fibonacci numbers.

ii. **(7 Points)** Using your result from part (i), prove that \( F_n \) and \( F_{n+1} \) are relatively prime for every \( n \in \mathbb{N} \). Remember that 0 ∈ \( \mathbb{N} \), and you can assume that 0 and 1 are relatively prime.
(Extra space for your answer to Problem Four, Part (ii), if you need it.)