This exam is closed-book and closed-computer. You may have a double-sided, 8.5” × 11” sheet of notes with you when you take this exam. You may not have any other notes with you during the exam. You may not use any electronic devices (laptops, cell phones, etc.) during the course of this exam. Please write all of your solutions on this physical copy of the exam.

You are welcome to cite results from the problem sets or lectures on this exam. Just tell us what you're citing and where you're citing it from. However, please do not cite results that are beyond the scope of what we've covered in CS103.

On the actual exam, there'd be space here for you to write your name and sign a statement saying you abide by the Honor Code. We're not collecting or grading this exam (though you're welcome to step outside and chat with us about it when you're done!) and this exam doesn't provide any extra credit, so we've opted to skip that boilerplate.

You have three hours to complete this practice midterm. There are 36 total points. This practice midterm is purely optional and will not directly impact your grade in CS103, but we hope that you find it to be a useful way to prepare for the exam. You may find it useful to read through all the questions to get a sense of what this practice midterm contains before you begin.

You have three hours to complete this exam. There are 48 total points.

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Problem One: Equivalence Relations

(12 Points)

(Midterm Exam, Winter 2018)

On Problem Set Three, you explored equivalence relations and a number of their properties. This problem explores the interaction of equivalence relations with some terminology that, up to this point, we've more typically associated with functions.

Let’s begin with a new definition. A binary relation $R$ over a set $A$ is called surjective if the following statement is true about $R$:

$$\forall b \in A. \exists a \in A. aRb.$$  

This definition is closely related to the definition of surjectivity for functions, hence the name. Prove that a binary relation $R$ over a set $A$ is an equivalence relation if and only if $R$ is surjective, symmetric, and transitive.
(Extra space for your answer to Problem One, if you need it.)
Problem Two: Functions and Sets  
(Midterm Exam, Winter 2018)

On Problem Sets Three, Four, and Five, you explored mathematical functions, their properties, and their applications. This question explores set theory, which you studied in Problem Set One and Problem Set Two, and its interaction with functions.

Imagine you have a function $f : A \rightarrow B$ from some set $A$ to some set $B$. We can use $f$ to construct a new function called the lift of $f$, denoted $\text{lift}_f$, from $\mathcal{P}(A)$ to $\mathcal{P}(B)$. Specifically $\text{lift}_f : \mathcal{P}(A) \rightarrow \mathcal{P}(B)$ is defined as follows:

$$\text{lift}_f(S) = \{ y \mid \exists x \in S. f(x) = y \}$$

In other words, the function $\text{lift}_f(S)$ takes as input an element $S$ of $\mathcal{P}(A)$ (that is, a subset of $A$) and produces an element of $\mathcal{P}(B)$ (that is, a subset of $B$) formed by applying $f$ to each element of $S$.

i. (5 Points) Let $A$ and $B$ be sets. Prove that if $f : A \rightarrow B$ is not injective, then $\text{lift}_f$ is not injective.

Feel free to use this space for scratch work. There’s room to write your answer on the next page of this exam.
(Extra space for your answer to Problem Two, Part (i), if you need it.)
As a refresher from the previous page, if \( f : A \to B \) is a function, the \textit{lift of} \( f \), denoted \( \text{lift}_f \), is a function \( \text{lift}_f : \wp(A) \to \wp(B) \) defined as follows:

\[
\text{lift}_f(S) = \{ y \mid \exists x \in S. f(x) = y \}
\]

ii. \textbf{(7 Points)} Let \( A \) and \( B \) be sets. Prove that if \( f : A \to B \) is injective, then \( \text{lift}_f \) is injective.

Feel free to use this space for scratch work. There’s room to write your answer on the next page of this exam.
(Extra space for your answer to Problem Two, Part (ii), if you need it.)
Problem Three: Strict Orders

(Midterm Exam, Winter 2018)

On Problem Set Three, Problem Set Four, and Problem Set Five, you explored different properties of strict order relations and of functions between sets. This problem is designed to let you show us what you’ve learned about these types of structures.

Let’s begin with a new definition. If \( R_1 \) is a binary relation over a set \( A_1 \) and \( R_2 \) is a binary relation over a set \( A_2 \), then an embedding of \( R_1 \) in \( R_2 \) is a function \( f: A_1 \rightarrow A_2 \) such that

\[
\forall a \in A_1. \forall b \in A_1. (aR_1 b \leftrightarrow f(a) R_2 f(b)).
\]

If there’s an embedding of a relation \( R_1 \) in a relation \( R_2 \), we say that \( R_1 \) can be embedded in \( R_2 \).

i. (6 Points) Let \( R_1 \) be a binary relation over a set \( A_1 \) and let \( R_2 \) be a strict order over some set \( A_2 \). Prove that if \( R_1 \) can be embedded in \( R_2 \), then \( R_1 \) is a strict order.

Feel free to use the space below for scratch work. There’s room to write your answer to this problem on the next page of this exam.
(Extra space for your answer to Problem Three, Part (i), if you need it.)
As a refresher, if \( R_1 \) is a binary relation over a set \( A_1 \) and \( R_2 \) is a binary relation over a set \( A_2 \), then an embedding of \( R_1 \) in \( R_2 \) is a function \( f : A_1 \to A_2 \) such that

\[
\forall a \in A_1, \forall b \in A_1. \ (aR_1 b \Leftrightarrow f(a) R_2 f(b)).
\]

If there’s an embedding of a relation \( R_1 \) in a relation \( R_2 \), we say that \( R_1 \) can be embedded in \( R_2 \).

ii. (6 Points) In the indicated space below, draw diagrams of two binary relations \( R_1 \) and \( R_2 \) (which may or may not be over the same set) and an embedding \( f \) of \( R_1 \) in \( R_2 \) such that

- \( R_1 \) is a strict order, but
- \( R_2 \) is not a strict order.

Then, briefly justify your answer. There’s space on the next page of this exam for scratch work.

Draw relation \( R_1 \) here.

Draw relation \( R_2 \) here.

Draw the embedding \( f : A_1 \to A_2 \) by drawing arrows from each element of the domain to its corresponding element of the codomain.

Briefly justify your answer here:
(Scratch space for your answer to Problem Three, Part (ii), if you need it.)
Problem Four: Graphs and Induction  
(Midterm Exam, Winter 2018) 

(12 Points)

On Problem Set Four, you explored different properties of graphs and how to write proofs about them. On Problem Set Five, you practiced writing proofs by induction. This problem is designed to let you demonstrate what you’ve learned in the course of doing so.

Let’s begin with a new definition. A triangle-free graph is one that contains no triangles (as a refresher, a triangle is a collection of three mutually adjacent nodes). That is, if you pick three distinct nodes in the graph, some two of them will not be adjacent.

Prove by induction that if $G$ is a triangle-free graph with $2n$ nodes (for some natural number $n$), then $G$ has at most $n^2$ edges. Some hints:

- Remember that $(n + 1)^2 = n^2 + 2n + 1$.
- What happens if you pick a pair of adjacent nodes in a triangle-free graph and delete them? Think about trying to count up how many edges there are in the graph after you do this.

Feel free to use this page for scratch work. There’s room to write your answer to this problem on the next page of this exam.
(Extra space for your answer to Problem Four, if you need it.)