Mathematical Logic
Part Three
The Aristotelian Forms

“All As are Bs”
∀x. (A(x) → B(x))

“Some As are Bs”
∃x. (A(x) ∧ B(x))

“No As are Bs”
∀x. (A(x) → ¬B(x))

“Some As aren’t Bs”
∃x. (A(x) ∧ ¬B(x))

It is worth committing these patterns to memory. We’ll be using them throughout the day and they form the backbone of many first-order logic translations.
Combining Quantifiers

• Most interesting statements in first-order logic require a combination of quantifiers.

• Example: “Everyone loves someone else.”
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- For every person, there is some person who isn't them.
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For every person, there is some person who isn't them that they love.
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• Example: “There is someone everyone else loves.”
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There is some person who everyone who isn't them loves.
For Comparison

\[ \forall p. (\text{Person}(p) \rightarrow \exists q. (\text{Person}(q) \land p \neq q \land \text{Loves}(p, q))) \]

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\[ \exists p. (\text{Person}(p) \land \forall q. (\text{Person}(q) \land p \neq q \rightarrow \text{Loves}(q, p))) \]

There is some person who everyone who isn't them loves.
Everyone Loves Someone Else
Everyone Loves Someone Else

No one here is universally loved.
There is Someone Everyone Else Loves
There is Someone Everyone Else Loves

This person does not love anyone else.
Everyone Loves Someone Else *and*
There is Someone Everyone Else Loves
∀p. (Person(p) → ∃q. (Person(q) ∧ p ≠ q ∧ Loves(p, q)))

For every person, there is some person who isn't them that they love.

∧

∃p. (Person(p) ∧ ∀q. (Person(q) ∧ p ≠ q → Loves(q, p)))

There is some person who everyone who isn't them loves.
Quantifier Ordering

• The statement

$$\forall x. \exists y. P(x, y)$$

means “for any choice of $x$, there's some choice of $y$ where $P(x, y)$ is true.”

• The choice of $y$ can be different every time and can depend on $x$. 
Quantifier Ordering

• The statement \( \exists x. \forall y. P(x, y) \)
  means “there is some \( x \) where for any choice of \( y \), we get that \( P(x, y) \) is true.”

• Since the inner part has to work for any choice of \( y \), this places a lot of constraints on what \( x \) can be.
Order matters when mixing existential and universal quantifiers!
Set Translations
Using the predicates

- \( Set(S) \), which states that \( S \) is a set, and
- \( x \in y \), which states that \( x \) is an element of \( y \),

write a sentence in first-order logic that means “the empty set exists.””
Using the predicates

- \textit{Set}(S), which states that \( S \) is a set, and
- \( x \in y \), which states that \( x \) is an element of \( y \),

write a sentence in first-order logic that means \textit{“the empty set exists.”}“
How many of the following first-order logic statements are correct translations of “the empty set exists”?

\[
\exists S. \ (\text{Set}(S) \land \neg \exists x. \ x \in S)
\]
\[
\exists S. \ (\text{Set}(S) \land \exists x. \ x \notin S)
\]
\[
\exists S. \ (\text{Set}(S) \land \neg \forall x. \ x \in S)
\]
\[
\exists S. \ (\text{Set}(S) \land \forall x. \ x \notin S)
\]

Answer at PollEv.com/cs103 or text CS103 to 22333 once to join, then 0, 1, 2, 3, or 4.
The empty set exists.
There is some set $S$ that is empty.
\exists S. (Set(S) \land \text{S is empty.})
\( \exists S. (\text{Set}(S) \land \text{there are no elements in } S) \)
\exists S. (\text{Set}(S) \land \neg \text{there is an element in } S )
\( \exists S. \ (Set(S) \land \neg \text{there is an element } x \text{ in } S) \)
\[ \exists S. \ (\text{Set}(S) \land \neg \exists x. \ x \in S) \]
\( \exists S. (Set(S) \land \neg \exists x. x \in S) \)
\exists S. (Set(S) \land \neg \exists x. x \in S)

\exists S. (Set(S) \land
\hspace{1cm} there are no elements in S)
\hspace{1cm} )
∀S. (Set(S) ∧ ¬∃x. x ∈ S)

∀S. (Set(S) ∧
    every object does not belong to S
 )
\[ \exists S. (Set(S) \land \neg\exists x. x \in S) \]

\[ \exists S. (Set(S) \land \\
\quad \text{every object } x \text{ does not belong to } S \\
) \]
\[ \exists S. (\text{Set}(S) \land \neg \exists x. x \in S) \]

\[ \exists S. (\text{Set}(S) \land \forall x. x \notin S) \]
\exists S. (Set(S) \land \neg \exists x. \ x \in S)

\exists S. (Set(S) \land \forall x. \ x \notin S)
Both of these translations are correct. Just like in propositional logic, there are many different equivalent ways of expressing the same statement in first-order logic.
Mechanics: Negating Statements
Which of the following is the negation of the statement \( \forall x. \exists y. Loves(x, y) \)?

A. \( \forall x. \forall y. \neg Loves(x, y) \)
B. \( \forall x. \exists y. \neg Loves(x, y) \)
C. \( \exists x. \forall y. \neg Loves(x, y) \)
D. \( \exists x. \exists y. \neg Loves(x, y) \)
E. None of these.
F. Two or more of these.

Answer at PollEv.com/cs103 or text CS103 to 22333 once to join, then A, B, C, D, E, or F.
Before we can answer this poll question, let’s review some first-order logic negation mechanics.
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Negating First-Order Statements

• Use the equivalences

\[
\neg \forall x. A \equiv \exists x. \neg A \\
\neg \exists x. A \equiv \forall x. \neg A
\]

to negate quantifiers.

• Mechanically:
  • Push the negation across the quantifier.
  • Change the quantifier from \( \forall \) to \( \exists \) or vice-versa.

• Use techniques from propositional logic to negate connectives.
Back to our poll question:
Which of the following is the negation of the statement \( \forall x. \exists y. Loves(x, y) \)?

\[ \forall x. \exists y. Loves(x, y) \]
(“Everyone loves someone.”)

\[ \neg \forall x. \exists y. Loves(x, y) \]
\[ \exists x. \neg \exists y. Loves(x, y) \]
\[ \exists x. \forall y. \neg Loves(x, y) \]
(“There's someone who doesn't love anyone.”)
Two Useful Equivalences

- The following equivalences are useful when negating statements in first-order logic:
  \[ \neg(p \land q) \equiv p \rightarrow \neg q \]
  \[ \neg(p \rightarrow q) \equiv p \land \neg q \]

- These identities are useful when negating statements involving quantifiers.
  - \( \land \) is used in existentially-quantified statements.
  - \( \rightarrow \) is used in universally-quantified statements.

- When pushing negations across quantifiers, we strongly recommend using the above equivalences to keep \( \rightarrow \) with \( \forall \) and \( \land \) with \( \exists \).
Negating Quantifiers

• What is the negation of the following statement, which says “there is a cute puppy”?

  \[ \exists x. (\text{Puppy}(x) \land \text{Cute}(x)) \]

• We can obtain it as follows:

  \[ \neg \exists x. (\text{Puppy}(x) \land \text{Cute}(x)) \]

  \[ \forall x. \neg (\text{Puppy}(x) \land \text{Cute}(x)) \]

  \[ \forall x. (\text{Puppy}(x) \rightarrow \neg \text{Cute}(x)) \]

• This says “no puppy is cute.”

• Do you see why this is the negation of the original statement from both an intuitive and formal perspective?
\[ \exists S. \ (Set(S) \land \forall x. \neg (x \in S)) \]
(“There is a set with no elements.”)

\[ \neg \exists S. \ (Set(S) \land \forall x. \neg (x \in S)) \]

\[ \forall S. \ (Set(S) \rightarrow \neg \forall x. \neg (x \in S)) \]

\[ \forall S. \ (Set(S) \rightarrow \exists x. \neg \neg (x \in S)) \]

\[ \forall S. \ (Set(S) \rightarrow \exists x. \ x \in S) \]
(“Every set contains at least one element.”)
These two statements are *not* negations of one another. Can you explain why?

\[ \exists S. \ (\text{Set}(S) \land \forall x. \neg(x \in S)) \]  
("There is a set that doesn't contain anything")

\[ \forall S. \ (\text{Set}(S) \land \exists x. \ (x \in S)) \]  
("Everything is a set that contains something")

Remember: \( \forall \) usually goes with \( \rightarrow \), not \( \land \)
Restricted Quantifiers
Quantifying Over Sets

• The notation

\[ \forall x \in S. \ P(x) \]

means “for any element \( x \) of set \( S \), \( P(x) \) holds.” (It’s vacuously true if \( S \) is empty.)

• The notation

\[ \exists x \in S. \ P(x) \]

means “there is an element \( x \) of set \( S \) where \( P(x) \) holds.” (It’s false if \( S \) is empty.)
Expressing Uniqueness
Using the predicate

- \( GOAT(p) \), which states that \( p \) is a \( GOAT \),

To do: write a sentence in first-order logic that means “there is only one GOAT.”
There is only one GOAT.
Something is a GOAT, and nothing else is.
Some thing \( p \) is a GOAT, and nothing else is.
Some thing $p$ is a GOAT, and nothing besides $p$ is a GOAT
\[\exists p. \ (GOAT(p) \land \text{nothing besides } p \text{ is a GOAT}.)\]
∃p. (\textit{GOAT}(p) \land \\
\textit{anything that isn't } p \textit{ isn't a level} \\
)
\( \exists p. \ (\text{GOAT}(p) \land \text{anything } x \text{ that isn't } p \text{ isn't a GOAT}) \)
\[ \exists p. \ (\text{GOAT}(p) \land \\
\forall x. \ (x \neq p \rightarrow x \text{ isn't a GOAT})) \]
$\exists p. \ (GOAT(p) \land \ \forall x. \ (x \neq p \rightarrow \neg GOAT(x))$
\[ \exists p. (GOAT(p) \land \forall x. (x \neq p \rightarrow \neg GOAT(x))) \]
\exists p. (GOAT(p) \land \\
\forall x. (GOAT(x) \rightarrow x = p) \\
)
Expressing Uniqueness

• To express the idea that there is exactly one object with some property, we write that
  • there exists at least one object with that property, and that
  • there are no other objects with that property.
Using the predicates

- \textit{Set}(S), which states that \( S \) is a set, and
- \( x \in y \), which states that \( x \) is an element of \( y \),

write a sentence in first-order logic that means “two sets are equal if and only if they contain the same elements.”
Two sets are equal if and only if they have the same elements.
Any two sets are equal if and only if they have the same elements.
Any two sets S and T are equal if and only if they have the same elements.
∀S. (Set(S) →
   ∀T. (Set(T) →
       S and T are equal if and only if they have the same elements.
   )
)
)
∀S. (Set(S) →
    ∀T. (Set(T) →
        (S = T if and only if they have the same elements.))
    )
)
∀S. (Set(S) ↪
  ∀T. (Set(T) ↪
      (S = T ↱ \text{they have the same elements}.))
  )
)
∀S. (Set(S) →
    ∀T. (Set(T) →
        (S = T ↔ S and T have the same elements.))
    )
)
∀S. (Set(S) \rightarrow
    ∀T. (Set(T) \rightarrow
        (S = T \leftrightarrow \text{every element of } S \text{ is an element of } T \text{ and }
        \text{vice-versa})
    ))
)
∀S. (Set(S) \to
  ∀T. (Set(T) \to
    (S = T \leftrightarrow x \text{ is an element of } S \text{ if and only if } x \text{ is an element of } T))
  )
)
\[ \forall S. \ (Set(S) \rightarrow \forall T. \ (Set(T) \rightarrow \ (S = T \leftrightarrow \forall x. \ (x \in S \leftrightarrow x \in T))) \]
∀S. (Set(S) →
    ∀T. (Set(T) →
        (S = T ↔ \forall x. (x \in S \leftrightarrow x \in T)))
    )
)
∀S. (Set(S) →
    ∀T. (Set(T) →
      (S = T ↔ ∀x. (x ∈ S ↔ x ∈ T)))
    )
  )

You sometimes see the universal quantifier pair with the ↔ connective. This is especially common when talking about sets because two sets are equal when they have precisely the same elements.
∀S. (Set(S) →
   ∀T. (Set(T) →
       (S = T ↔ ∀x. (x ∈ S ↔ x ∈ T)))
   )
)
Next Time

- **Binary Relations**
  - How do we model connections between objects?
- **Equivalence Relations**
  - How do we model the idea that objects can be grouped into clusters?
- **First-Order Definitions**
  - Where does first-order logic come into all of this?
- **Proofs with Definitions**
  - How does first-order logic interact with proofs?