Binary Relations
Part II
Outline for Today

• Proving an Equivalence Relation
  • A proof that ~ is an equivalence relation

• Properties of Equivalence Relations
  • What’s so special about those three rules?

• Cyclic Property
  • How it relates to our other three properties, and equivalence relations
\( \forall a \in A. \ aRa \)

\( \forall a \in A. \ \forall b \in A. \ (aRb \rightarrow bRa) \)

\( \forall a \in A. \ \forall b \in A. \ \forall c \in A. \ (aRb \land bRc \rightarrow aRc) \)
Equivalence Relation Proofs

• Let's suppose you've found a binary relation $R$ over a set $A$ and want to prove that it's an equivalence relation.

• How exactly would you go about doing this?
An Example Relation

- Consider the binary relation ~ defined over the set \( \mathbb{Z} \):
  \[
  a \sim b \quad \text{if} \quad a + b \text{ is even}
  \]
- Some examples:
  \[
  0 \sim 4 \quad 1 \sim 9 \quad 2 \sim 6 \quad 5 \sim 5
  \]
- Turns out, this is an equivalence relation! Let's see how to prove it.

We can binary relations by giving a rule, like this:

\[
\text{\( a \sim b \quad \text{if} \quad \text{some property of } a \text{ and } b \text{ holds} \)}
\]

*This is the general template for defining a relation.* Although we're using “if” rather than “iff” here, the two above statements are definitionally equivalent. For a variety of reasons, definitions are often introduced with “if” rather than “iff.” Check the “Mathematical Vocabulary” handout for details.
What properties must $\sim$ have to be an equivalence relation?

*Reflexivity*

*Symmetry*

*Transitivity*

Let's prove each property independently.
Lemma 1: The binary relation ~ is reflexive.
Lemma 1: The binary relation \( \sim \) is reflexive.

Proof:
$a \sim b$ if $a + b$ is even

**Lemma 1:** The binary relation $\sim$ is reflexive.

**Proof:**

What is the formal definition of reflexivity?
The binary relation $\sim$ is reflexive.

Proof:

What is the formal definition of reflexivity?

$$\forall a \in \mathbb{Z}. a \sim a$$
Lemma 1: The binary relation $\sim$ is reflexive.

Proof:

What is the formal definition of reflexivity?

$\forall a \in \mathbb{Z}. \ a \sim a$

Therefore, we'll choose an arbitrary integer $a$, then go prove that $a \sim a$. 
Lemma 1: The binary relation \( \sim \) is reflexive.

Proof:

What is the formal definition of reflexivity?

\[ \forall a \in \mathbb{Z}. \ a \sim a \]

ASSUME: Choose an arbitrary integer \( a \).

WANT TO SHOW: We want to show that \( a \sim a \).
Lemma 1: The binary relation ~ is reflexive.

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What is the formal definition of reflexivity?

\[ \forall a \in \mathbb{Z}. \ a \sim a \]

ASSUME: Choose an arbitrary integer \( a \).

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Lemma 1: The binary relation $\sim$ is reflexive.

Proof: Consider an arbitrary $a \in \mathbb{Z}$. We need to prove that $a \sim a$. 

$$a \sim b \text{ if } a+b \text{ is even}$$
$a \sim b$ if $a+b$ is even

**Lemma 1:** The binary relation $\sim$ is reflexive.

**Proof:** Consider an arbitrary $a \in \mathbb{Z}$. We need to prove that $a \sim a$. From the definition of the $\sim$ relation, this means that we need to prove that $a + a$ is even.

To see this, notice that $a + a = 2a$, so the sum $a + a$ can be written as $2k$ for some integer $k$ (namely, $a$), so $a + a$ is even. Therefore, $a \sim a$ holds, as required. ■
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Lemma 2: The binary relation \( \sim \) is symmetric.

\( a \sim b \) if \( a + b \) is even
Lemma 2: The binary relation ~ is symmetric.

Which of the following works best as the opening ("assume" part) of this proof?

A. Consider any integers $a$ and $b$. We will prove $a \sim b$ and $b \sim a$.
B. Pick $\forall a \in \mathbb{Z}$ and $\forall b \in \mathbb{Z}$. We will prove $a \sim b \rightarrow b \sim a$.
C. Consider any integers $a$ and $b$ where $a \sim b$ and $b \sim a$.
D. Consider any integer $a$ where $a \sim a$.
E. The relation ~ is symmetric if for any $a, b \in \mathbb{Z}$, we have $a \sim b \rightarrow b \sim a$.
F. Consider any integers $a$ and $b$ where $a \sim b$. We will prove $b \sim a$.

Answer at PollEv.com/cs103 or text CS103 to 22333 once to join, then A, B, C, D, E, or F.
Lemma 2: The binary relation $\sim$ is symmetric.

Proof:

If $a \sim b$, then $a + b$ is even. Because $a + b = b + a$, this means that $b + a$ is even. Since $b + a$ is even, we know that $b \sim a$, as required. ■
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Proof:

What is the formal definition of symmetry?
Lemma 2: The binary relation \( \sim \) is symmetric.

Proof:

What is the formal definition of symmetry?

\[ \forall a \in \mathbb{Z}. \forall b \in \mathbb{Z}. (a \sim b \rightarrow b \sim a) \]
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Therefore, we'll choose arbitrary integers $a$ and $b$ where $a \sim b$, then prove that $b \sim a$. 
$a \sim b$ if $a+b$ is even

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Which of the following works best as the opening ("assume" part) of this proof?

A. Consider any integers \( a \) and \( b \). We will prove \( a \sim b \) and \( b \sim a \).
B. Pick \( \forall a \in \mathbb{Z} \) and \( \forall b \in \mathbb{Z} \). We will prove \( a \sim b \rightarrow b \sim a \).
C. Consider any integers \( a \) and \( b \) where \( a \sim b \) and \( b \sim a \).
D. Consider any integer \( a \) where \( a \sim a \).
E. The relation \( \sim \) is symmetric if for any \( a, b \in \mathbb{Z} \), we have \( a \sim b \rightarrow b \sim a \).
F. Consider any integers \( a \) and \( b \) where \( a \sim b \). We will prove \( b \sim a \).
Lemma 2: The binary relation \( \sim \) is symmetric.

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Since \( a \sim b \), we know that \( a+b \) is even.
Lemma 2: The binary relation \( \sim \) is symmetric.

Proof: Consider any integers \( a \) and \( b \) where \( a \sim b \). We need to show that \( b \sim a \).

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Proof: Consider any integers $a$ and $b$ where $a \sim b$. We need to show that $b \sim a$.

Since $a \sim b$, we know that $a+b$ is even. Because $a+b = b+a$, this means that $b+a$ is even. Since $b+a$ is even, we know that $b \sim a$, as required.
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Lemma 3: The binary relation $\sim$ is transitive.
Lemma 3: The binary relation ~ is transitive.

Proof:

\[ a \sim b \quad \text{if} \quad a + b \text{ is even} \]
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What is the formal definition of transitivity?
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\[
\forall a \in \mathbb{Z} . \; \forall b \in \mathbb{Z} . \; \forall c \in \mathbb{Z} . \; (a \sim b \land b \sim c \rightarrow a \sim c)
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Lemma 3: The binary relation \( \sim \) is transitive.

Proof: Consider arbitrary integers \( a \), \( b \), and \( c \) where \( a \sim b \) and \( b \sim c \). We need to prove that \( a \sim c \), meaning that we need to show that \( a + c \) is even. Since \( a \sim b \) and \( b \sim c \), we know that \( a \sim b \) and \( b \sim c \) are even. This means there are integers \( k \) and \( m \) where \( a + b = 2k \) and \( b + c = 2m \). Notice that \((a + b) + (b + c) = 2k + 2m \). Rearranging, we see that \( a + c + 2b = 2k + 2m \), so \( a + c = 2k + 2m - 2b = 2(k + m - b) \). So there is an integer \( r \), namely \( k + m - b \), such that \( a + c = 2r \). Thus \( a + c \) is even, so \( a \sim c \), as required. ■

What is the formal definition of transitivity?

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\forall a \in \mathbb{Z}. \forall b \in \mathbb{Z}. \forall c \in \mathbb{Z}. \ (a \sim b \land b \sim c \rightarrow a \sim c)
\]

Therefore, we'll choose arbitrary integers \( a, b, \) and \( c \) where \( a \sim b \) and \( b \sim c \), then prove that \( a \sim c \).
Lemma 3: The binary relation ~ is transitive.

Proof: Consider arbitrary integers $a$, $b$ and $c$ where $a \sim b$ and $b \sim c$. If $a + b$ is even, then $a \sim c$. This can be shown as follows:

Since $a \sim b$ and $b \sim c$, we know that $a + b$ and $b + c$ are even. This means there are integers $k$ and $m$ where $a + b = 2k$ and $b + c = 2m$. Notice that $(a + b) + (b + c) = 2k + 2m$. Rearranging, we see that $a + b + 2b = 2k + 2m$, so $a + b = 2(k + m - b)$. Since $k + m - b$ is an integer, there is an integer $r$, namely $k + m - b$, such that $a + b = 2r$. Thus $a + b$ is even, so $a \sim c$, as required. ■
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$$(a+b) + (b+c) = 2k + 2m.$$
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Rearranging, we see that

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a+c + 2b = 2k + 2m,
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so there is an integer \( r \), namely \( k + m - b \), such that \( a+c = 2r \). Thus \( a+c \) is even, so \( a \sim c \), as required. ■
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So there is an integer $r$, namely $k + m - b$, such that $a + c = 2r$. Thus $a + c$ is even, so $a \sim c$, as required.
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An Observation
Lemma 1: The binary relation ~ is reflexive.

Proof: Consider an arbitrary $a \in \mathbb{Z}$. We need to prove that $a \sim a$. From the definition of the ~ relation, this means that we need to prove that $a + a$ is even.

To see this, notice that $a + a = 2a$, so the sum $a + a$ can be written as $2k$ for some integer $k$ (namely, $a$), so $a + a$ is even. Therefore, $a \sim a$ holds, as required. ■
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Proof: Consider any integers \( a \) and \( b \) where \( a \sim b \). We need to show that \( b \sim a \).

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\[ a \sim b \quad \text{if} \quad a+b \text{ is even} \]

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a+c + 2b = 2k + 2m,
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so

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a+c = 2k + 2m - 2b = 2(k+m-b).
\]

So there is an integer \( r \), namely \( k+m-b \), such that \( a+c = 2r \). Thus \( a+c \) is even.

The formal definition of transitivity is given in first-order logic, but **this proof does not contain any first-order logic symbols**!
First-Order Logic and Proofs

• First-order logic is an excellent tool for giving formal definitions to key terms.

• While first-order logic *guides* the structure of proofs, it is *exceedingly rare* to see first-order logic in written proofs.

• Follow the example of these proofs:
  
  • Use the first-order logic definitions to identify your "*assume*" and "*want to show*" parts of the proof.
  
  • Write the proof in plain English using the conventions we set up in the first week of the class.
Properties of Equivalence Relations
$xTy$ if $x$ and $y$ have the same color
\[ xRy \quad \text{if} \quad x \text{ and } y \text{ have the same shape} \]
Equivalence Classes

- Given an equivalence relation $R$ over a set $A$, for any $x \in A$, the equivalence class of $x$ is the set

$$[x]_R = \{ y \in A \mid xRy \}$$

- Intuitively, the set $[x]_R$ contains all elements of $A$ that are related to $x$ by relation $R$. 
$xRy$ if $x$ and $y$ have the same shape
Recall equivalence classes definition:

$$[x]_R = \{ y \in A \mid xRy \}$$

**How many different names** could we use to refer to the equivalence class on the right (with the suns)?

Answer at PollEv.com/cs103 or text **CS103** to **22333** once to join, then 0, 1, 2, 3, or 4.
$xRy$ if $x$ and $y$ have the same shape
$x R y$ if $x$ and $y$ have the same shape
$xRy$ if $x$ and $y$ have the same shape
The Fundamental Theorem of Equivalence Relations: Let $R$ be an equivalence relation over a set $A$. Then every element $a \in A$ belongs to exactly one equivalence class of $R$. 
How’d We Get Here?

- We discovered equivalence relations by thinking about *partitions* of a set of elements.
- We saw that if we had a binary relation that tells us whether two elements are in the same group, it had to be reflexive, symmetric, and transitive.
- The FToER says that, in some sense, these rules precisely capture what it means to be a partition.
- **Question:** What’s so special about these three rules?
\[ \forall a \in A. \ aRa \]

\[ \forall a \in A. \ \forall b \in A. \ (aRb \rightarrow bRa) \]

\[ \forall a \in A. \ \forall b \in A. \ \forall c \in A. \ (aRb \land bRc \rightarrow aRc) \]
A new rule that must be true:

Relation this person holds: “Are these two things in the same partition?” for some mystery partition.
\( aRb \land bRc \rightarrow cRa \)
∀a ∈ A. ∀b ∈ A. ∀c ∈ A. (aRb ∧ bRc → cRa)
\[ \forall a \in A. \ \forall b \in A. \ \forall c \in A. \ (aRb \land bRc \rightarrow cRa) \]

A binary relation with this property is called **cyclic**.
**Theorem:** A binary relation $R$ over a set $A$ is an equivalence relation if and only if it is reflexive and cyclic.
Theorem: A binary relation $R$ over a set $A$ is an equivalence relation if and only if it is reflexive and cyclic.
Lemma 1: If $R$ is an equivalence relation over a set $A$, then $R$ is reflexive and cyclic.

Lemma 2: If $R$ is a binary relation over a set $A$ that is reflexive and cyclic, then $R$ is an equivalence relation.
**Lemma 1:** If $R$ is an equivalence relation over a set $A$, then $R$ is reflexive and cyclic.

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  $R$ is symmetric.  
  $R$ is transitive. | $R$ is reflexive.  
  $R$ is cyclic. |
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Lemma 1: If \( R \) is an equivalence relation over a set \( A \), then \( R \) is reflexive and cyclic.

### What We’re Assuming
- \( R \) is an equivalence relation.
  - \( R \) is reflexive.
  - \( R \) is symmetric.
  - \( R \) is transitive.

### What We Need To Show
- If \( aRb \) and \( bRc \), then \( cRa \).
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![Diagram](image-url)
**Lemma 1:** If $R$ is an equivalence relation over a set $A$, then $R$ is reflexive and cyclic.

**What We’re Assuming**

- $R$ is an equivalence relation.
  - $R$ is reflexive.
  - $R$ is symmetric.
  - $R$ is transitive.

**What We Need To Show**

- If $aRb$ and $bRc$, then $cRa$. 

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You write the next sentence!
What is our assumption?

Answer at PollEv.com/cs103 or text CS103 to 22333 once to join, then your sentence.
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Notice how the first few sentences of this proof mirror the structure of what needs to be proved. We’re just following the templates from the first week of class!
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Notice how this setup mirrors the first-order definition of cyclicity:

$$\forall a \in A. \forall b \in A. \forall c \in A. (aRb \land bRc \rightarrow cRa)$$

When writing proofs about terms with first-order definitions, it’s critical to call back to those definitions!
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![Diagram](attachment:attachment.png)
**Lemma 2:** If $R$ is a binary relation over a set $A$ that is reflexive and cyclic, then $R$ is an equivalence relation.

**What We’re Assuming**

- $R$ is reflexive.
  - $\forall x \in A. \ xRx$
- $R$ is cyclic.
  - $xRy \land yRz \rightarrow zRx$

**What We Need To Show**

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![Diagram showing reflexive and cyclic properties]
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![Diagram showing the cyclic relationships between elements a, b, and c.](image)
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Proof: Let \( R \) be an arbitrary binary relation over a set \( A \) that is cyclic and reflexive. We need to prove that \( R \) is an equivalence relation. To do so, we need to show that \( R \) is reflexive, symmetric, and transitive. Since we already know by assumption that \( R \) is reflexive, we just need to show that \( R \) is symmetric and transitive.

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Notice how this setup mirrors the first-order definition of transitivity:

$$\forall a \in A. \forall b \in A. \forall c \in A. (aRb \land bRc \to aRc)$$

When writing proofs about terms with first-order definitions, it’s critical to call back to those definitions!
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Next Time

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  • How do we model transformations in a mathematical sense?

• **Domains and Codomains**
  • Type theory meets mathematics!

• **Injections, Surjections, and Bijections**
  • Three special classes of functions.