Binary Relations
Part II
Outline for Today

- **Finish from Last Time**
  - Pt. 3 of our proof that \( \sim \) is an equivalence relation
- **Properties of Equivalence Relations**
  - What’s so special about those three rules?
- **Strict Orders**
  - A different type of mathematical structure
- **Hasse Diagrams**
  - How to visualize rankings
Finish from Last Time
\forall a \in A. \ aRa

\forall a \in A. \ \forall b \in A. \ (aRb \rightarrow bRa)

\forall a \in A. \ \forall b \in A. \ \forall c \in A. \ (aRb \land bRc \rightarrow aRc)
\[ a \sim b \quad \text{if} \quad a+b \text{ is even} \]

**Lemma 1:** The binary relation \( \sim \) is reflexive.

**Proof:** Consider an arbitrary \( a \in \mathbb{Z} \). We need to prove that \( a \sim a \). From the definition of the \( \sim \) relation, this means that we need to prove that \( a + a \) is even.

To see this, notice that \( a + a = 2a \), so the sum \( a + a \) can be written as \( 2k \) for some integer \( k \) (namely, \( a \)), so \( a + a \) is even. Therefore, \( a \sim a \) holds, as required. ■
Lemma 2: The binary relation $\sim$ is symmetric.

Proof: Consider any integers $a$ and $b$ where $a \sim b$. We need to show that $b \sim a$.

Since $a \sim b$, we know that $a + b$ is even. Because $a + b = b + a$, this means that $b + a$ is even. Since $b + a$ is even, we know that $b \sim a$, as required. ■
New Stuff!
∀a ∈ A. aRa

∀a ∈ A. ∀b ∈ A. (aRb → bRa)

∀a ∈ A. ∀b ∈ A. ∀c ∈ A. (aRb ∧ bRc → aRc)
Lemma 3: The binary relation \( \sim \) is transitive.

\[ a \sim b \quad \text{if} \quad a + b \text{ is even} \]
Lemma 3: The binary relation ~ is transitive.

Proof: Consider arbitrary integers $a$, $b$, and $c$ where $a \sim b$ and $b \sim c$. We need to prove that $a \sim c$, meaning that we need to show that $a + c$ is even.

Since $a \sim b$ and $b \sim c$, we know that $a \sim b$ and $b \sim c$ are even. This means there are integers $k$ and $m$ where $a + b = 2k$ and $b + c = 2m$. Notice that $(a + b) + (b + c) = 2k + 2m$.

Rearranging, we see that $a + c + 2b = 2k + 2m$, so $a + c = 2k + 2m - 2b = 2(k + m - b)$. So there is an integer $r$, namely $k + m - b$, such that $a + c = 2r$. Thus $a + c$ is even, so $a \sim c$, as required. ■
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Proof:

$a \sim b$ if $a + b$ is even
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What is the formal definition of transitivity?
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What is the formal definition of transitivity?

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\forall a \in \mathbb{Z}. \forall b \in \mathbb{Z}. \forall c \in \mathbb{Z}. (a \sim b \land b \sim c \rightarrow a \sim c)
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Rearranging, we see that $a + c + 2b = 2k + 2m$, so $a + c = 2k + 2m - 2b = 2(k + m - b)$.

So there is an integer $r$, namely $k + m - b$, such that $a + c = 2r$. Thus $a + c$ is even, so $a \sim c$, as required. ■

What is the formal definition of transitivity?

$$\forall a \in \mathbb{Z}. \forall b \in \mathbb{Z}. \forall c \in \mathbb{Z}. (a \sim b \land b \sim c \rightarrow a \sim c)$$

Therefore, we'll choose arbitrary integers $a$, $b$, and $c$ where $a \sim b$ and $b \sim c$, then prove that $a \sim c$. 
If $a + b$ is even, then $a \sim b$. 

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\[ \text{Since } a \sim b \text{ and } b \sim c, \text{ we know that } a \sim b \text{ and } b \sim c \text{ are even.} \]

This means there are integers \( k \) and \( m \) where \( a+b = 2k \) and \( b+c = 2m \). Notice that \((a+b) + (b+c) = 2k + 2m\).

Rearranging, we see that \( a+c + 2b = 2k + 2m \), so \( a+c = 2(k+m) - 2b = 2(r) \), where \( r = k+m-b \).

So there is an integer \( r \), namely \( k+m-b \), such that \( a+c = 2r \). Thus \( a+c \) is even, so \( a \sim c \), as required. \( \blacksquare \)
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Rearranging, we see that

$$a + c + 2b = 2k + 2m,$$

so

$$a + c = 2k + 2m - 2b = 2(k + m - b).$$
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So there is an integer \( r \), namely \( k+m-b \), such that \( a+c = 2r \). Thus \( a+c \) is even.

The formal definition of transitivity is given in first-order logic, but this proof does not contain any first-order logic symbols!
First-Order Logic and Proofs

• First-order logic is an excellent tool for giving formal definitions to key terms.

• While first-order logic *guides* the structure of proofs, it is *exceedingly rare* to see first-order logic in written proofs.

• Follow the example of these proofs:
  • Use the FOL definitions to determine what to assume and what to prove.
  • Write the proof in plain English using the conventions we set up in the first week of the class.

• *Please, please, please, please, please, please, please internalize the contents of this slide!*
\[ \forall a \in A. \ aRa \]

\[ \forall a \in A. \ \forall b \in A. \ (aRb \rightarrow bRa) \]

\[ \forall a \in A. \ \forall b \in A. \ \forall c \in A. \ (aRb \land bRc \rightarrow aRc) \]
Properties of Equivalence Relations
$xR_y \quad \text{if} \quad x \text{ and } y \text{ have the same shape}$
$xTy \quad \text{if} \quad x \text{ and } y \text{ have the same color}$
Equivalence Classes

• Given an equivalence relation $R$ over a set $A$, for any $x \in A$, the **equivalence class of $x$** is the set

  $$[x]_R = \{ y \in A \mid xRy \}$$

• Intuitively, the set $[x]_R$ contains all elements of $A$ that are related to $x$ by relation $R$. 
\[ xRy \quad \text{if} \quad x \text{ and } y \text{ have the same shape} \]
The Fundamental Theorem of Equivalence Relations: Let $R$ be an equivalence relation over a set $A$. Then every element $a \in A$ belongs to exactly one equivalence class of $R$. 
\[ x R y \quad \text{if} \quad x \text{ and } y \text{ have the same shape} \]
How’d We Get Here?

- We discovered equivalence relations by thinking about partitions of a set of elements.
- We saw that if we had a binary relation that tells us whether two elements are in the same group, it had to be reflexive, symmetric, and transitive.
- The FToER says that, in some sense, these rules precisely capture what it means to be a partition.
- **Question**: What’s so special about these three rules?
The question we are asking the sage: “Are these two in the same equivalence class?”
aRb \land bRc \rightarrow cRa
\[ \forall a \in A. \forall b \in A. \forall c \in A. (aRb \land bRc \rightarrow cRa) \]
∀a ∈ A. ∀b ∈ A. ∀c ∈ A. (aRb ∧ bRc → cRa)

A binary relation with this property is called \textit{cyclic}. 
Let $R$ be the relation depicted here. How many of the following claims are true?

- $R$ is reflexive.
- $R$ is symmetric.
- $R$ is transitive.
- $R$ is an equivalence relation.

Answer at PollEv.com/cs103 or text CS103 to 22333 once to join, then 0, 1, 2, 3, or 4.

∀a ∈ A. ∀b ∈ A. ∀c ∈ A. (aRb ∧ bRc → cRa)
**Theorem:** A binary relation $R$ over a set $A$ is an equivalence relation if and only if it is reflexive and cyclic.
**Theorem:** A binary relation $R$ over a set $A$ is an equivalence relation *if and only if* it is reflexive and cyclic.
Lemma 1: If $R$ is an equivalence relation over a set $A$, then $R$ is reflexive and cyclic.

Lemma 2: If $R$ is a binary relation over a set $A$ that is reflexive and cyclic, then $R$ is an equivalence relation.
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Proof: Let $R$ be an arbitrary equivalence relation over some set $A$. 

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Since $R$ is an equivalence relation, we know that $R$ is reflexive, symmetric, and transitive. Consequently, we already know that $R$ is reflexive, so we only need to show that $R$ is cyclic.

To prove that $R$ is cyclic, consider any arbitrary $a, b, c \in A$ where $aRb$ and $bRc$. We need to prove that $cRa$ holds.

Since $R$ is transitive, from $aRb$ and $bRc$ we see that $aRc$.

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Proof: Let $R$ be an arbitrary equivalence relation over some set $A$. We need to prove that $R$ is reflexive and cyclic. Since $R$ is an equivalence relation, we know that $R$ is reflexive, so we only need to show that $R$ is cyclic. To prove that $R$ is cyclic, consider any arbitrary $a, b, c \in A$ where $aRb$ and $bRc$. We need to prove that $cRa$ holds. Since $R$ is transitive, from $aRb$ and $bRc$ we see that $aRc$. Then, since $R$ is symmetric, from $aRc$ we see that $cRa$, which is what we needed to prove. ■

Notice how this setup mirrors the first-order definition of cyclicity:

$$\forall a \in A. \forall b \in A. \forall c \in A. (aRb \land bRc \rightarrow cRa)$$

When writing proofs about terms with first-order definitions, it’s critical to call back to those definitions!
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Diagram:

- $a$ (yellow smiley)
- $b$ (blue smiley)

Connected by an arrow pointing from $a$ to $b$.
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![Diagram showing cyclic relation](attachment:image.png)
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Lemma 2: If $R$ is a binary relation over a set $A$ that is cyclic and reflexive, then $R$ is an equivalence relation.

Proof: Let $R$ be an arbitrary binary relation over a set $A$ that is cyclic and reflexive. We need to prove that $R$ is an equivalence relation. To do so, we need to show that $R$ is reflexive, symmetric, and transitive. Since we already know by assumption that $R$ is reflexive, we just need to show that $R$ is symmetric and transitive.

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Notice how this setup mirrors the first-order definition of symmetry:

$$\forall a \in A. \forall b \in A. (aRb \rightarrow bRa)$$

When writing proofs about terms with first-order definitions, it’s critical to call back to those definitions!
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Notice how this setup mirrors the first-order definition of transitivity:

$$\forall a \in A. \forall b \in A. \forall c \in A. \ (aRb \land bRc \rightarrow aRc)$$

When writing proofs about terms with first-order definitions, it's critical to call back to those definitions!
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Refining Your Proofwriting

- When writing proofs about terms with formal definitions, you **must** call back to those definitions.
  - Use the first-order definition to see what you’ll assume and what you’ll need to prove.
- When writing proofs about terms with formal definitions, you **must not** include any first-order logic in your proofs.
  - Although you won’t use any FOL *notation* in your proofs, your proof implicitly calls back to the FOL definitions.
- You’ll get a lot of practice with this on Problem Set Three. If you have any questions about how to do this properly, please feel free to ask on Piazza or stop by office hours!
Prerequisite Structures
The CS Core

Systems

CS106B
Programming Abstractions

CS107
Computer Organization and Systems

CS110
Principles of Computer Systems

Theory

CS103
Mathematical Foundations of Computing

CS109
Intro to Probability for Computer Scientists

CS161
Design and Analysis of Algorithms
Pancakes

Everyone's got a pancake recipe. This one comes from Food Wishes (http://foodwishes.blogspot.com/2011/08/grandma-kellys-good-old-fashioned.html).

Ingredients

- 1 1/2 cups all-purpose flour
- 3 1/2 tsp baking powder
- 1 tsp salt
- 1 tbsp sugar
- 1 1/4 cup milk
- 1 egg
- 3 tbsp butter, melted

Directions

1. Sift the dry ingredients together.
2. Stir in the butter, egg, and milk. Whisk together to form the batter.
3. Heat a large pan or griddle on medium-high heat. Add some oil.
4. Make pancakes one at a time using 1/4 cup batter each. They're ready to flip when the centers of the pancakes start to bubble.
Measure Flour
Measure Sugar
Measure Baking Pwdr
Measure Salt

Beat Egg

Combine Dry Ingredients

Melt Butter

Measure Milk

Heat Griddle

Add Wet Ingredients

Oil Griddle

Make Pancakes

Serve Pancakes
Relations and Prerequisites

• Let's imagine that we have a prerequisite structure with no circular dependencies.

• We can think about a binary relation $R$ where $aRb$ means

  \textit{“a must happen before b”}

• What properties of $R$ could we deduce just from this?
$aRa$

\[
aRb \land bRc \rightarrow aRc
\]

\[
aRb \rightarrow bRa
\]
∀a ∈ A. aRa

∀a ∈ A. ∀b ∈ A. ∀c ∈ A. (aRb ∧ bRc → aRc)

∀a ∈ A. ∀b ∈ A. (aRb → bRa)
∀a ∈ A. aRa

Transitivity

∀a ∈ A. ∀b ∈ A. (aRb → bRa)
∀a ∈ A. aRa

Transitivity

∀a ∈ A. ∀b ∈ A. (aRb → bRa)
Irreflexivity

• Some relations *never* hold from any element to itself.

• As an example, $x \not\leq x$ for any $x$.

• Relations of this sort are called *irreflexive*.

• Formally speaking, a binary relation $R$ over a set $A$ is irreflexive if the following first-order logic statement is true about $R$:

$$\forall a \in A. aRa$$

(“*No element is related to itself.*”)
Irreflexivity Visualized

∀a ∈ A. aRa
(“No element is related to itself.”)
Let $R$ be the relation depicted here. How many of the following claims are true?

- $R$ is reflexive.
- $R$ is not reflexive.
- $R$ is irreflexive.
- $R$ is not irreflexive.

Answer at PollEv.com/cs103 or text CS103 to 22333 once to join, then 0, 1, 2, 3, or 4.
Is this relation reflexive?
∀a ∈ A. aRa
(“Every element is related to itself.”)
$\forall a \in A. \ aRa$

("Every element is related to itself.")
∀a ∈ A. aRa

(“Every element is related to itself.”)
Is this relation irreflexive?
∀a ∈ A. aRa
(“No element is related to itself.”)
∀a ∈ A. aRa
(“No element is related to itself.”)
$\forall a \in A. \ aRa$

("No element is related to itself.")
Reflexivity and Irreflexivity

• Reflexivity and irreflexivity are not opposites!
• Here's the definition of reflexivity:
  \[ \forall a \in A. \ aRa \]
• What is the negation of the above statement?
  \[ \exists a \in A. \ a \not\in Ra \]
• What is the definition of irreflexivity?
  \[ \forall a \in A. \ aRa \]
∀a ∈ A. aRa

Transitivity

∀a ∈ A. ∀b ∈ A. (aRb → bRa)
Irreflexivity

∀a ∈ A. ∀b ∈ A. (aRb → bRb)

Transitivity

∀a ∈ A. ∀b ∈ A. (aRb → bRa)
Irreflexivity

\[ \forall a \in A. \forall b \in A. (a R b \rightarrow b \not R a) \]
Asymmetry

• In some relations, the relative order of the objects can never be reversed.
• As an example, if $x < y$, then $y \not< x$.
• These relations are called asymmetric.
• Formally: a binary relation $R$ over a set $A$ is called asymmetric if the following first-order logic statement is true about $R$:

$$\forall a \in A. \forall b \in A. (aRs \rightarrow bR\neg a)$$

(“If $a$ relates to $b$, then $b$ does not relate to $a$.”)
∀a ∈ A. ∀b ∈ A. (aRb → bRa)
(“If a relates to b, then b does not relate to a.”)
Question to Ponder: Are symmetry and asymmetry opposites of one another?
Irreflexivity

∀a ∈ A. ∀b ∈ A. (aRb → bϕa)

Transitivity
Irreflexivity

Transitivity

Asymmetry
Strict Orders

- A strict order is a relation that is irreflexive, asymmetric and transitive.
- Some examples:
  - $x < y$.
  - $a$ can run faster than $b$.
  - $A \subset B$ (that is, $A \subseteq B$ and $A \neq B$).
- Strict orders are useful for representing prerequisite structures and have applications in complexity theory (measuring notions of relative hardness) and algorithms (searching and sorting).
Drawing Strict Orders
<table>
<thead>
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<th>Gold</th>
<th>Silver</th>
<th>Bronze</th>
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</tr>
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<td>6</td>
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<td>2</td>
</tr>
</tbody>
</table>
\((g_1, s_1, b_1) \, R \, (g_2, s_2, b_2)\) \quad \text{if} \quad g_1 < g_2 \, \land \, s_1 < s_2 \, \land \, b_1 < b_2
\[(g_1, s_1, b_1) \text{ } R \text{ } (g_2, s_2, b_2) \text{ if } g_1 < g_2 \land s_1 < s_2 \land b_1 < b_2\]
$g_1 < g_2 \land s_1 < s_2 \land b_1 < b_2$
\[(g_1, s_1, b_1) \ R \ (g_2, s_2, b_2) \quad \text{if} \quad g_1 < g_2 \ \land \ s_1 < s_2 \ \land \ b_1 < b_2\]
More Medals

(g₁, s₁, b₁) R (g₂, s₂, b₂) if g₁ < g₂ ∧ s₁ < s₂ ∧ b₁ < b₂

Fewer Medals
More Medals

$$(g_1, s_1, b_1) \, R \, (g_2, s_2, b_2) \quad \text{if} \quad g_1 < g_2 \, \land \, s_1 < s_2 \, \land \, b_1 < b_2$$

Fewer Medals
\[(g_1, s_1, b_1) \, R \, (g_2, s_2, b_2) \quad \text{if} \quad g_1 < g_2 \land s_1 < s_2 \land b_1 < b_2\]
Hasse Diagrams

- A **Hasse diagram** is a graphical representation of a strict order.
- Elements are drawn from bottom-to-top.
- No self loops are drawn, and none are needed! By **irreflexivity** we know they shouldn’t be there.
- Higher elements are bigger than lower elements: by **asymmetry**, the edges can only go in one direction.
- No redundant edges: by **transitivity**, we can infer the missing edges.
If \[ 5g_1 + 3s_1 + b_1 \geq 5g_2 + 3s_2 + b_2 \] then \((g_1, s_1, b_1) T (g_2, s_2, b_2)\).
If

\[ 5g_1 + 3s_1 + b_1 < 5g_2 + 3s_2 + b_2 \]
5g₁ + 3s₁ + b₁ \leq 5g₂ + 3s₂ + b₂

if

(g₁, s₁, b₁) \in T (g₂, s₂, b₂)
\[(g_1, s_1, b_1) \cup (g_2, s_2, b_2) \text{ if } g_1 + s_1 + b_1 < g_2 + s_2 + b_2\]
(g₁, s₁, b₁) U (g₂, s₂, b₂)

if

g₁ + s₁ + b₁ < g₂ + s₂ + b₂
(g₁, s₁, b₁) ∪ (g₂, s₂, b₂)

if

g₁ + s₁ + b₁ < g₂ + s₂ + b₂
Hasse Artichokes

\[ xRy \quad \text{if} \quad x \quad \text{must be eaten before} \quad y \]
Hasse Artichokes

xRy if x must be eaten before y
The Meta Strict Order

Strict Order

Irreflexivity

Asymmetry

Transitivity

Equivalence Relation

Reflexivity

Symmetry

Question to ponder: why is this line here?

$aRb$ if $a$ is less specific than $b$
Next Time

- **Functions**
  - How do we model transformations in a mathematical sense?

- **Domains and Codomains**
  - Type theory meets mathematics!

- **Injections, Surjections, and Bijections**
  - Three special classes of functions.