Functions
What is a function?
Functions, High-School Edition
\[ f(x) = x^4 - 5x^2 + 4 \]
$f(x) = \frac{x^2 + 4x - 9}{x^2 + 10x + 21}$
Functions, High-School Edition

• In high school, functions are usually given as objects of the form

\[ f(x) = \frac{x^3 + 3x^2 + 15x + 7}{1 - x^{137}} \]

• What does a function do?
  • It takes in as input a real number.
  • It outputs a real number
  • ... except when there are vertical asymptotes or other discontinuities, in which case the function doesn't output anything.
Functions, CS Edition
```c
int flipUntil(int n) {
    int numHeads = 0;
    int numTries = 0;

    while (numHeads < n) {
        if (randomBoolean()) numHeads++;
        numTries++;
    }

    return numTries;
}
```
In programming, functions
- might take in inputs,
- might return values,
- might have side effects,
- might never return anything,
- might crash, and
- might return different values when called multiple times.
What's Common?

- Although high-school math functions and CS functions are pretty different, they have two key aspects in common:
  - They take in inputs.
  - They produce outputs.
- In math, we like to keep things easy, so that's pretty much how we're going to define a function.
Rough Idea of a Function:

A function is an object $f$ that takes in an input and produces exactly one output.

(This is not a complete definition – we'll revisit this in a bit.)
High School versus CS Functions

• In high school, functions usually were given by a rule:
  \[ f(x) = 4x + 15 \]

• In CS, functions are usually given by code:
  ```c
  int factorial(int n) {
    int result = 1;
    for (int i = 1; i <= n; i++) {
      result *= i;
    }
    return result;
  }
  ```

• What sorts of functions are we going to allow from a mathematical perspective?
... but also ...
\[ f(x) = x^2 + 3x - 15 \]
$$f(n) = \begin{cases}  
-n/2 & \text{if } n \text{ is even} \\
(n+1)/2 & \text{otherwise} 
\end{cases}$$

Functions like these are called *piecewise functions*. 
To define a function, you will typically either
- draw a picture, or
- give a rule for determining the output.
In mathematics, functions are *deterministic*. That is, given the same input, a function must always produce the same output.

The following is a perfectly valid piece of C++ code, but it’s not a valid function under our definition:

```cpp
int randomNumber(int numOutcomes) {
    return rand() % numOutcomes;
}
```
One Challenge
\[ f(x) = x^2 + 2x + 5 \]
\[ f(x) = x^2 + 2x + 5 \]

\[ f(3) = 3^2 + 3 \cdot 2 + 5 = 20 \]
\[ f(x) = x^2 + 2x + 5 \]

\[ f(3) = 3^2 + 3 \cdot 2 + 5 = 20 \]

\[ f(0) = 0^2 + 0 \cdot 2 + 5 = 5 \]
\[ f(x) = x^2 + 2x + 5 \]

\[ f(3) = 3^2 + 3 \cdot 2 + 5 = 20 \]
\[ f(0) = 0^2 + 0 \cdot 2 + 5 = 5 \]

\[ f(\text{ Pikachu }) = \ldots ? \]
\[ f( \text{Pikachu}) = \text{Coro} \]
\[ f(137) = \ldots ? \]
We need to make sure we can't apply functions to meaningless inputs.
Domains and Codomains

- Every function $f$ has two sets associated with it: its domain and its codomain.
- A function $f$ can only be applied to elements of its domain. For any $x$ in the domain, $f(x)$ belongs to the codomain.

The function must be defined for every element of the domain.

The output of the function must always be in the codomain, but not all elements of the codomain must be produced as outputs.
Domains and Codomains

- Every function $f$ has two sets associated with it: its **domain** and its **codomain**.
- A function $f$ can only be applied to elements of its domain. For any $x$ in the domain, $f(x)$ belongs to the codomain.

```java
double absoluteValueOf(double x) {
    if (x >= 0) {
        return x;
    } else {
        return -x;
    }
}
```

The **domain** of this function is $\mathbb{R}$. Any real number can be provided as input.

The **codomain** of this function is $\mathbb{R}$. Everything produced is a real number, but not all real numbers can be produced.
Domains and Codomains

- If $f$ is a function whose domain is $A$ and whose codomain is $B$, we write $f : A \rightarrow B$.
- Think of this like a “function prototype” in C++.
The Official Rules for Functions

• Formally speaking, we say that $f : A \rightarrow B$ if the following two rules hold.

• First, $f$ must be obey its domain/codomain rules:
  \[ \forall a \in A. \exists b \in B. \ f(a) = b \]
  (“Every input in A maps to some output in B.”)

• Second, $f$ must be deterministic:
  \[ \forall a_1 \in A. \forall a_2 \in A. \ (a_1 = a_2 \rightarrow f(a_1) = f(a_2)) \]
  ("Equal inputs produce equal outputs.”)

• If you’re ever curious about whether something is a function, look back at these rules and check! For example:
  • Can a function have an empty domain?
  • Can a function with a nonempty domain have an empty codomain?
Defining Functions

• Typically, we specify a function by describing a rule that maps every element of the domain to some element of the codomain.

• Examples:
  • $f(n) = n + 1$, where $f : \mathbb{Z} \to \mathbb{Z}$
  • $f(x) = \sin x$, where $f : \mathbb{R} \to \mathbb{R}$
  • $f(x) = \lfloor x \rfloor$, where $f : \mathbb{R} \to \mathbb{Z}$

• Notice that we're giving both a rule and the domain/codomain.
Defining Functions

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• Notice that we're giving both a rule and the domain/codomain.
Defining Functions

Typically, we specify a function by describing a rule that maps every element of the domain to some element of the codomain.

Examples:

- $f(n) = n + 1$, where $f : \mathbb{Z} \rightarrow \mathbb{Z}$
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- $f(x) = \lceil x \rceil$, where $f : \mathbb{R} \rightarrow \mathbb{Z}$

Notice that we're giving both a rule and the domain/codomain.

This is the ceiling function – the smallest integer greater than or equal to $x$.

For example, $\lceil 1 \rceil = 1$, $\lceil 1.37 \rceil = 2$, and $\lceil 3.14 \rceil = 4$. 
Is This a Function From $A$ to $B$?

**A**

- Stanford
- Berkeley
- Michigan
- Arkansas

**B**

- Cardinal
- Blue
- Gold
- White
Is This a Function From $A$ to $B$?

- California
- New York
- Delaware
- Washington DC

- Sacramento
- Dover
- Albany

A \to B
Is This a Function From $A$ to $B$?
Combining Functions
\[ h : \text{People} \rightarrow \text{Prices} \]

\[ h(x) = g(f(x)) \]
Function Composition

• Suppose that we have two functions $f : A \rightarrow B$ and $g : B \rightarrow C$.

• Notice that the codomain of $f$ is the domain of $g$. This means that we can use outputs from $f$ as inputs to $g$. 

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$\begin{align*}
\text{x} \quad \rightarrow \quad f \quad \rightarrow \quad f(x) \quad \rightarrow \quad g \quad \rightarrow \quad g(f(x))
\end{align*}$
Function Composition

• Suppose that we have two functions \( f : A \rightarrow B \) and \( g : B \rightarrow C \).

• The \textit{composition of \( f \) and \( g \)}, denoted \( g \circ f \), is a function where
  \begin{itemize}
  \item \( g \circ f : A \rightarrow C \), and
  \item \( (g \circ f)(x) = g(f(x)) \).
  \end{itemize}

• A few things to notice:
  \begin{itemize}
  \item The domain of \( g \circ f \) is the domain of \( f \). Its codomain is the codomain of \( g \).
  \item Even though the composition is written \( g \circ f \), when evaluating \( (g \circ f)(x) \), the function \( f \) is evaluated first.
  \end{itemize}

The name of the function is \( g \circ f \). When we apply it to an input \( x \), we write \( (g \circ f)(x) \). I don't know why, but that's what we do.
Time-Out for Announcements!
Problem Set One Feedback

• Problem Set One grades have been posted
  • If you haven’t already, please review the feedback we’ve left for you as soon as possible, as well as the solution set.
  • We’re happy to answer any questions about specific comments in office hours or on Piazza.
  • If you believe we’ve made a grading error, see the Regrade Policies handout for instructions on how to submit a regrade.
Problem Set Three

• Problem Set Two was due today at 3:00PM.
  • Want to use late days? Submit the assignment by Sunday at 3:00PM.

• Problem Set Three goes out today.
  • The checkpoint is due on Monday at 3:00PM.
  • The remaining problems are due Friday at 3:00PM.
  • Play around with binary relations, functions, their properties, and their applications!

• As usual, *feel free to ask questions*!
  • Ask on Piazza!
  • Stop by office hours!
Back to CS103!
Special Types of Functions
Injective Functions

- A function \( f : A \to B \) is called **injective** (or **one-to-one**) if the following statement is true about \( f \):

\[
\forall a_1 \in A. \forall a_2 \in A. (a_1 \neq a_2 \rightarrow f(a_1) \neq f(a_2))
\]

(“If the inputs are different, the outputs are different.”)

- The following first-order definition is equivalent and is often useful in proofs.

\[
\forall a_1 \in A. \forall a_2 \in A. (f(a_1) = f(a_2) \rightarrow a_1 = a_2)
\]

(“If the outputs are the same, the inputs are the same.”)

- A function with this property is called an **injection**.

- How does this compare to our second rule for functions?
Injective Functions

**Theorem:** Let \( f : \mathbb{N} \rightarrow \mathbb{N} \) be defined as \( f(n) = 2n + 7 \). Then \( f \) is injective.
Injective Functions

**Theorem:** Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be defined as $f(n) = 2n + 7$. Then $f$ is injective.

**Proof:**
Injective Functions

**Theorem:** Let \( f : \mathbb{N} \to \mathbb{N} \) be defined as \( f(n) = 2n + 7 \). Then \( f \) is injective.

**Proof:**

How many of the following are correct ways of starting off this proof?

Consider any \( n_1, n_2 \in \mathbb{N} \) where \( n_1 = n_2 \). We will prove that \( f(n_1) = f(n_2) \).

Consider any \( n_1, n_2 \in \mathbb{N} \) where \( n_1 \neq n_2 \). We will prove that \( f(n_1) \neq f(n_2) \).

Consider any \( n_1, n_2 \in \mathbb{N} \) where \( f(n_1) = f(n_2) \). We will prove that \( n_1 = n_2 \).

Consider any \( n_1, n_2 \in \mathbb{N} \) where \( f(n_1) \neq f(n_2) \). We will prove that \( n_1 \neq n_2 \).
Injective Functions

**Theorem:** Let $f : \mathbb{N} \to \mathbb{N}$ be defined as $f(n) = 2n + 7$. Then $f$ is injective.

**Proof:**

What does it mean for the function $f$ to be injective?
Injective Functions

**Theorem:** Let \( f : \mathbb{N} \rightarrow \mathbb{N} \) be defined as \( f(n) = 2n + 7 \). Then \( f \) is injective.

**Proof:**

What does it mean for the function \( f \) to be injective?

\[
\forall n_1 \in \mathbb{N}. \forall n_2 \in \mathbb{N}. \ ( f(n_1) = f(n_2) \rightarrow n_1 = n_2 )
\]

\[
\forall n_1 \in \mathbb{N}. \forall n_2 \in \mathbb{N}. \ ( n_1 \neq n_2 \rightarrow f(n_1) \neq f(n_2) )
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\forall n_1 \in \mathbb{N}. \forall n_2 \in \mathbb{N}. ( n_1 \neq n_2 \rightarrow f(n_1) \neq f(n_2) )
\]

Therefore, we'll pick arbitrary \( n_1, n_2 \in \mathbb{N} \) where \( f(n_1) = f(n_2) \), then prove that \( n_1 = n_2 \).
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\[ \forall n_1 \in \mathbb{N}. \forall n_2 \in \mathbb{N}. \ ( f(n_1) = f(n_2) \rightarrow n_1 = n_2 ) \]

\[ \forall n_1 \in \mathbb{N}. \forall n_2 \in \mathbb{N}. \ ( n_1 \neq n_2 \rightarrow f(n_1) \neq f(n_2) ) \]

Therefore, we'll pick arbitrary \( n_1, n_2 \in \mathbb{N} \) where \( f(n_1) = f(n_2) \), then prove that \( n_1 = n_2 \).
Injective Functions

**Theorem:** Let $f : \mathbb{N} \to \mathbb{N}$ be defined as $f(n) = 2n + 7$. Then $f$ is injective.

**Proof:**

What does it mean for the function $f$ to be injective?

$$\forall n_1 \in \mathbb{N}. \forall n_2 \in \mathbb{N}. \left( f(n_1) = f(n_2) \rightarrow n_1 = n_2 \right)$$

$$\forall n_1 \in \mathbb{N}. \forall n_2 \in \mathbb{N}. \left( n_1 \neq n_2 \rightarrow f(n_1) \neq f(n_2) \right)$$

Therefore, we'll pick arbitrary $n_1, n_2 \in \mathbb{N}$ where $f(n_1) = f(n_2)$, then prove that $n_1 = n_2$. 
Theorem: Let $f : \mathbb{N} \to \mathbb{N}$ be defined as $f(n) = 2n + 7$. Then $f$ is injective.

Proof: Consider any $n_1, n_2 \in \mathbb{N}$ where $f(n_1) = f(n_2)$. We will prove that $n_1 = n_2$. 

...
Injective Functions

**Theorem:** Let $f : \mathbb{N} \to \mathbb{N}$ be defined as $f(n) = 2n + 7$. Then $f$ is injective.

**Proof:** Consider any $n_1, n_2 \in \mathbb{N}$ where $f(n_1) = f(n_2)$. We will prove that $n_1 = n_2$.

Since $f(n_1) = f(n_2)$, we see that

$$2n_1 + 7 = 2n_2 + 7.$$
Injective Functions

\textbf{Theorem: } Let \( f : \mathbb{N} \to \mathbb{N} \) be defined as \( f(n) = 2n + 7 \). Then \( f \) is injective.

\textbf{Proof: } Consider any \( n_1, n_2 \in \mathbb{N} \) where \( f(n_1) = f(n_2) \). We will prove that \( n_1 = n_2 \).

Since \( f(n_1) = f(n_2) \), we see that
\[
2n_1 + 7 = 2n_2 + 7.
\]
This in turn means that
\[
2n_1 = 2n_2
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Since $f(n_1) = f(n_2)$, we see that

$$2n_1 + 7 = 2n_2 + 7.$$

This in turn means that

$$2n_1 = 2n_2$$

so $n_1 = n_2$, as required.
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Since \( f(n_1) = f(n_2) \), we see that

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2n_1 + 7 = 2n_2 + 7.
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This in turn means that

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2n_1 = 2n_2
\]

so \( n_1 = n_2 \), as required. ■
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**Proof:** Consider any \( n_1, n_2 \in \mathbb{N} \) where \( f(n_1) = f(n_2) \). We will prove that \( n_1 = n_2 \).

**Good exercise:** Repeat this proof using the other definition of injectivity!

---

**How many of the following are correct ways of starting off this proof?**

Consider any \( n_1, n_2 \in \mathbb{N} \) where \( n_1 = n_2 \). We will prove that \( f(n_1) = f(n_2) \).
Consider any \( n_1, n_2 \in \mathbb{N} \) where \( n_1 \neq n_2 \). We will prove that \( f(n_1) \neq f(n_2) \).
Consider any \( n_1, n_2 \in \mathbb{N} \) where \( f(n_1) = f(n_2) \). We will prove that \( n_1 = n_2 \).
Consider any \( n_1, n_2 \in \mathbb{N} \) where \( f(n_1) \neq f(n_2) \). We will prove that \( n_1 \neq n_2 \).
Injective Functions

**Theorem:** Let $f : \mathbb{Z} \rightarrow \mathbb{N}$ be defined as $f(x) = x^4$. Then $f$ is not injective.
Injective Functions

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**Proof:**

Let $x_0 = -1$ and $x_1 = +1$. Then $f(x_0) = f(-1) = (-1)^4 = 1$ and $f(x_1) = f(1) = 1^4 = 1$, so $f(x_0) = f(x_1)$ even though $x_0 \neq x_1$, as required. ■
Injective Functions

Theorem: Let $f : \mathbb{Z} \to \mathbb{N}$ be defined as $f(x) = x^4$. Then $f$ is not injective.

Proof:

How many of the following are correct ways of starting off this proof?

- Assume for the sake of contradiction that $f$ is not injective.
- Assume for the sake of contradiction that there are integers $x_1$ and $x_2$ where $f(x_1) = f(x_2)$ but $x_1 \neq x_2$.
- Consider arbitrary integers $x_1$ and $x_2$ where $x_1 \neq x_2$. We will prove that $f(x_1) = f(x_2)$.
- Consider arbitrary integers $x_1$ and $x_2$ where $f(x_1) = f(x_2)$. We will prove that $x_1 \neq x_2$. 

Let $x_0 = -1$ and $x_1 = +1$. Then $f(x_0) = f(-1) = (-1)^4 = 1$ and $f(x_1) = f(1) = 1^4 = 1$, so $f(x_0) = f(x_1)$ even though $x_0 \neq x_1$, as required. ■
Injective Functions

Theorem: Let \( f : \mathbb{Z} \to \mathbb{N} \) be defined as \( f(x) = x^4 \). Then \( f \) is not injective.

Proof:

What does it mean for \( f \) to be injective?

\[ \forall x_0 \in \mathbb{Z}. \forall x_1 \in \mathbb{Z}. (x_0 \neq x_1 \rightarrow f(x_0) \neq f(x_1)) \]
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What is the negation of this statement?
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What is the negation of this statement?

\[
\neg \forall x_1 \in \mathbb{Z}. \forall x_2 \in \mathbb{Z}. (x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2))
\]

Therefore, we need to find \( x_0, x_1 \in \mathbb{Z} \) such that \( x_0 \neq x_1 \), but \( f(x_0) = f(x_1) \). Can we do that?
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**Theorem:** Let $f : \mathbb{Z} \to \mathbb{N}$ be defined as $f(x) = x^4$. Then $f$ is not injective.

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What does it mean for $f$ to be injective?

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What is the negation of this statement?

\[ \neg \forall x_1 \in \mathbb{Z}. \forall x_2 \in \mathbb{Z}. (x_1 \neq x_2 \to f(x_1) \neq f(x_2)) \]

\[ \exists x_1 \in \mathbb{Z}. \neg \forall x_2 \in \mathbb{Z}. (x_1 \neq x_2 \to f(x_1) \neq f(x_2)) \]
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Therefore, we need to find $x_0, x_1 \in \mathbb{Z}$ such that $x_0 \neq x_1$, but $f(x_0) = f(x_1)$. Can we do that?
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What does it mean for $f$ to be injective?

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What is the negation of this statement?

$$\neg \forall x_1 \in \mathbb{Z}. \forall x_2 \in \mathbb{Z}. (x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2))$$

$$\exists x_1 \in \mathbb{Z}. \neg \forall x_2 \in \mathbb{Z}. (x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2))$$

$$\exists x_1 \in \mathbb{Z}. \exists x_2 \in \mathbb{Z}. \neg (x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2))$$

Therefore, we need to find $x_0, x_1 \in \mathbb{Z}$ such that $x_0 \neq x_1$, but $f(x_0) = f(x_1)$. Can we do that?
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\exists x_1 \in \mathbb{Z}. \exists x_2 \in \mathbb{Z.} \neg (x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2))
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Therefore, we need to find \( x_0, x_1 \in \mathbb{Z} \) such that \( x_0 \neq x_1 \), but \( f(x_0) = f(x_1) \). Can we do that?
Injective Functions

**Theorem:** Let \( f : \mathbb{Z} \to \mathbb{N} \) be defined as \( f(x) = x^4 \). Then \( f \) is not injective.

**Proof:**

What does it mean for \( f \) to be injective?

\[
\forall x_1 \in \mathbb{Z}. \forall x_2 \in \mathbb{Z}. (x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2))
\]

What is the negation of this statement?

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Let \( x_1 = -1 \) and \( x_2 = +1 \).
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$$f(x_1) = f(-1) = (-1)^4 = 1$$
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$$f(x_1) = f(-1) = (-1)^4 = 1$$

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$$f(x_2) = f(1) = 1^4 = 1,$$

so $f(x_1) = f(x_2)$ even though $x_1 \neq x_2$, as required.
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\[
\begin{align*}
f(x_1) &= f(-1) = (-1)^4 = 1 \\
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Injective Functions

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---

How many of the following are correct ways of starting this proof?

1. Assume for the sake of contradiction that $f$ is not injective.
2. Assume for the sake of contradiction that there are integers $x_1$ and $x_2$ where $f(x_1) = f(x_2)$ but $x_1 \neq x_2$.
3. Consider arbitrary integers $x_1$ and $x_2$ where $x_1 \neq x_2$. We will prove that $f(x_1) = f(x_2)$.
4. Consider arbitrary integers $x_1$ and $x_2$ where $f(x_1) = f(x_2)$. We will prove that $x_1 \neq x_2$. 

---

Let $x_0 = -1$ and $x_1 = +1$. Then $f(x_0) = f(-1) = (-1)^4 = 1$ and $f(x_1) = f(1) = 1^4 = 1$, so $f(x_0) = f(x_1)$ even though $x_0 \neq x_1$, as required. ■
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$$f(x_1) = f(-1) = (-1)^4 = 1$$

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so $f(x_1) = f(x_2)$ even though $x_1 \neq x_2$, as required. ■
Injections and Composition
Injections and Composition

• **Theorem:** If \( f : A \to B \) is an injection and \( g : B \to C \) is an injection, then the function \( g \circ f : A \to C \) is an injection.

• Our goal will be to prove this result. To do so, we're going to have to call back to the formal definitions of injectivity and function composition.
**Theorem:** If $f : A \rightarrow B$ is an injection and $g : B \rightarrow C$ is an injection, then the function $g \circ f : A \rightarrow C$ is also an injection.

Proof: Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be arbitrary injections. We will prove that the function $g \circ f : A \rightarrow C$ is also injective.

To do so, consider any $a_1, a_2 \in A$ where $a_1 \neq a_2$. We will prove that $(g \circ f)(a_1) \neq (g \circ f)(a_2)$. Equivalently, we need to show that $g(f(a_1)) \neq g(f(a_2))$.

Since $f$ is injective and $a_1 \neq a_2$, we see that $f(a_1) \neq f(a_2)$.

Then, since $g$ is injective and $f(a_1) \neq f(a_2)$, we see that $g(f(a_1)) \neq g(f(a_2))$, as required.

\[\blacksquare\]
**Theorem:** If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is also an injection.

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**Proof:** Let $f : A \to B$ and $g : B \to C$ be arbitrary injections. We will prove that the function $g \circ f : A \to C$ is also injective.

There are two definitions of injectivity that we can use here:

1. $\forall a_1 \in A. \forall a_2 \in A. ((g \circ f)(a_1) = (g \circ f)(a_2) \to a_1 = a_2)$
2. $\forall a_1 \in A. \forall a_2 \in A. (a_1 \neq a_2 \to (g \circ f)(a_1) \neq (g \circ f)(a_2))$
**Theorem:** If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is also an injection.

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Therefore, we’ll choose an arbitrary \( a_1, a_2 \in A \) where \( a_1 \neq a_2 \), then prove that \( (g \circ f)(a_1) \neq (g \circ f)(a_2) \).
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How is $(g \circ f)(x)$ defined?
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$$(g \circ f)(x) = g(f(x))$$
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**How is \((g \circ f)(x)\) defined?**

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(g \circ f)(x) = g(f(x))
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So we need to prove that \( g(f(a_1)) \neq g(f(a_2)) \).
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Since $f$ is injective and $a_1 \neq a_2$, we see that $f(a_1) \neq f(a_2)$.

![Diagram](image-url)
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Great exercise: Repeat this proof using the other definition of injectivity.
Let’s take a five minute break!
Another Class of Functions
Surjective Functions

• A function \( f : A \to B \) is called \textit{surjective} (or \textit{onto}) if this first-order logic statement is true about \( f \):

\[
\forall b \in B. \exists a \in A. f(a) = b
\]

(“For every possible output, there's at least one possible input that produces it”)

• A function with this property is called a \textit{surjection}.

• How does this compare to our first rule of functions?
Surjective Functions

**Theorem:** Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined as $f(x) = x / 2$. Then $f(x)$ is surjective.
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**Proof:**
Surjective Functions

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What does it mean for $f$ to be surjective?
Surjective Functions

**Theorem:** Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = x / 2$. Then $f(x)$ is surjective.

**Proof:**

What does it mean for $f$ to be surjective?

$\forall y \in \mathbb{R}. \exists x \in \mathbb{R}. f(x) = y$
Surjective Functions

**Theorem:** Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined as $f(x) = x / 2$. Then $f(x)$ is surjective.

**Proof:**

What does it mean for $f$ to be surjective?

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Composing Surjections
**Theorem:** If \( f : A \rightarrow B \) is surjective and \( g : B \rightarrow C \) is surjective, then \( g \circ f : A \rightarrow C \) is also surjective.
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Therefore, we'll choose arbitrary $c \in C$ and prove that there is some $a \in A$ such that $(g \circ f)(a) = c$. 
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Consider any $c \in C$. 

\[ a \]
\[ A \]
\[ b \]
\[ B \]
\[ c \]
\[ C \]
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Injections and Surjections

• An injective function associates \textit{at most} one element of the domain with each element of the codomain.

• A surjective function associates \textit{at least} one element of the domain with each element of the codomain.

• What about functions that associate \textit{exactly one} element of the domain with each element of the codomain?
Bijections

• A function that associates each element of the codomain with a unique element of the domain is called bijective.
  • Such a function is a bijection.
• Formally, a bijection is a function that is both injective and surjective.
• Bijections are sometimes called one-to-one correspondences.
  • Not to be confused with “one-to-one functions.”
Bijections and Composition

• Suppose that $f : A \to B$ and $g : B \to C$ are bijections.

• Is $g \circ f$ necessarily a bijection?

• Yes!
  • Since both $f$ and $g$ are injective, we know that $g \circ f$ is injective.
  • Since both $f$ and $g$ are surjective, we know that $g \circ f$ is surjective.
  • Therefore, $g \circ f$ is a bijection.
Inverse Functions
Katniss Everdeen
Elsa
Hermione Granger
Inverse Functions

• In some cases, it's possible to “turn a function around.”

• Let $f : A \to B$ be a function. A function $f^{-1} : B \to A$ is called an inverse of $f$ if the following first-order logic statements are true about $f$ and $f^{-1}$

\[
\forall a \in A. (f^{-1}(f(a)) = a) \quad \forall b \in B. (f(f^{-1}(b)) = b)
\]

• In other words, if $f$ maps $a$ to $b$, then $f^{-1}$ maps $b$ back to $a$ and vice-versa.

• Not all functions have inverses (we just saw a few examples of functions with no inverses).

• If $f$ is a function that has an inverse, then we say that $f$ is invertible.
Inverse Functions

- **Theorem:** Let $f : A \rightarrow B$. Then $f$ is invertible if and only if $f$ is a bijection.
- These proofs are in the course reader. Feel free to check them out if you'd like!
- **Really cool observation:** Look at the formal definition of a function. Look at the rules for injectivity and surjectivity. Do you see why this result makes sense?
Let’s play a game!
Telephone Pictionary

1. Ninja fighting a pirate
2. Old man fights ninja kid
3. Beating an astronaut
4. Beating the Russians to the moon
5. 3 people dancing for a Taco
Telephone
Logictionary

1. Ninja fighting a pirate
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5. 3 people dancing for a Taco
How did it go?
Strategies to Keep in Mind

“All Ps are Qs.”
\( \forall x. (P(x) \rightarrow Q(x)) \)

“All Ps are Qs.”
\( \exists x. (P(x) \land Q(x)) \)

“No Ps are Qs.”
\( \forall x. (P(x) \rightarrow \neg Q(x)) \)

“No Ps are Qs.”
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- The \( \forall \) quantifier *usually* is paired with \( \rightarrow \).
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