Graph Theory
Part One
Outline for Today

- **Graphs and Digraphs**
  - Two fundamental mathematical structures.

- **Graphs Meet FOL**
  - Building visual intuitions.

- **Independent Sets and Vertex Covers**
  - Two structures in graphs.
Graphs and Digraphs
PANFLUTE FLOWCHART

- do you need one?
  - yes: no you don't
  - no: no panflute
What's in Common

• Each of these structures consists of
  • a collection of objects and
  • links between those objects.

• **Goal:** find a general framework for describing these objects and their properties.
A **graph** is a mathematical structure for representing relationships.
A graph is a mathematical structure for representing relationships.

A graph consists of a set of nodes (or vertices) connected by edges (or arcs).
A **graph** is a mathematical structure for representing relationships.

A graph consists of a set of **nodes** (or **vertices**) connected by **edges** (or **arcs**).
A graph is a mathematical structure for representing relationships.

A graph consists of a set of nodes (or vertices) connected by edges (or arcs).
Some graphs are directed.
Some graphs are *undirected*.
Graphs and Digraphs

- An **undirected graph** is one where edges link nodes, with no endpoint preferred over the other.
- A **directed graph** (or **digraph**) is one where edges have an associated direction.
- (There’s something called a **mixed graph** that allows for both, but they’re fairly uncommon and we won’t talk about them.)
- Unless specified otherwise:

  "Graph" means "undirected graph"
Formalizing Graphs

- How might we define a graph mathematically?
- We need to specify
  - what the nodes in the graph are, and
  - which edges are in the graph.
- The nodes can be pretty much anything.
- What about the edges?
Formalizing Graphs

- An **unordered pair** is a set \( \{a, b\} \) of two elements \( a \neq b \). (Remember that sets are unordered.)
  - For example, \( \{0, 1\} = \{1, 0\} \)
- An **undirected graph** is an ordered pair \( G = (V, E) \), where
  - \( V \) is a set of nodes, which can be anything, and
  - \( E \) is a set of edges, which are **unordered** pairs of nodes drawn from \( V \).
- A **directed graph** (or **digraph**) is an ordered pair \( G = (V, E) \), where
  - \( V \) is a set of nodes, which can be anything, and
  - \( E \) is a set of edges, which are **ordered** pairs of nodes drawn from \( V \).
- An **unordered pair** is a set \{a, b\} of two elements \(a \neq b\).
- An **undirected graph** is an ordered pair \(G = (V, E)\), where
  - \(V\) is a set of nodes, which can be anything, and
  - \(E\) is a set of edges, which are unordered pairs of nodes drawn from \(V\).

How many of these drawings are of valid undirected graphs?
Self-Loops

- An edge from a node to itself is called a self-loop.
- In (undirected) graphs, self-loops are generally not allowed.
  - Can you see how this follows from the definition?
- In digraphs, self-loops are generally allowed unless specified otherwise.
The Great Graph Gallery
Is this formula true about this graph?

\[ \forall u \in V. \exists v \in V. \{u, v\} \in E \]
Is this formula true about this graph?

\[ \exists u \in V. \ \forall v \in V. \ \{u, v\} \in E \]
Is this formula true about this graph?

\[ \exists u \in V. \ \forall v \in V. \ \{u, v\} \in E \]
Let's look at the negation!

$$\exists u \in V. \forall v \in V. \{u, v\} \in E$$
Let’s look at the negation!

$$\neg \exists u \in V. \forall v \in V. \{u, v\} \in E$$
Let’s look at the negation!

$$\forall u \in V. \neg \forall v \in V. \{u, v\} \in E$$
Let’s look at the negation!

\[ \forall u \in V. \exists v \in V. \neg(\{u, v\} \in E) \]
Let’s look at the negation!

∀u ∈ V. ∃v ∈ V. \{u, v\} ∉ E
Independent Sets and Vertex Covers
Two Motivating Problems
Place park rangers in these forest trails so that a hiker anywhere on a trail can see a park ranger.
Place park rangers in these forest trails so that a hiker anywhere on a trail can see a park ranger.
Place park rangers in these forest trails so that a hiker anywhere on a trail can see a park ranger.
Place park rangers in these forest trails so that a hiker anywhere on a trail can see a park ranger.
Place park rangers in these forest trails so that a hiker anywhere on a trail can see a park ranger.
Place park rangers in these forest trails so that a hiker anywhere on a trail can see a park ranger.
Choose at least one endpoint of each edge.
Choose at least one endpoint of each edge.
Choose at least one endpoint of each edge.
Choose at least one endpoint of each edge.
Vertex Covers

• Let $G = (V, E)$ be an undirected graph. A *vertex cover* of $G$ is a set $C \subseteq V$ such that the following statement is true:

$$\forall x \in V. \forall y \in V. \ (\{x, y\} \in E \rightarrow (x \in C \lor y \in C))$$

(“Every edge has at least one endpoint in $C$.”)

• Intuitively speaking, a vertex cover is a set formed by picking at least one endpoint of each edge in the graph.

• Vertex covers have applications to placing streetlights/benches/security guards, as well as in gene sequencing, machine learning, and combinatorics.
Set up nests for the California condor. Condors are territorial and won’t nest if they can see other condors.
Set up nests for the California condor. Condors are territorial and won’t nest if they can see other condors.
Set up nests for the California condor. Condors are territorial and won’t nest if they can see other condors.
Set up nests for the California condor. Condors are territorial and won’t nest if they can see other condors.
Set up nests for the California condor. Condors are territorial and won’t nest if they can see other condors.
Set up nests for the California condor. Condors are territorial and won’t nest if they can see other condors.
Choose a set of nodes, no two of which are adjacent.
Independent Sets

• If $G = (V, E)$ is an (undirected) graph, then an independent set in $G$ is a set $I \subseteq V$ such that

$$\forall u \in I. \forall v \in I. \{u, v\} \notin E.$$  

(“No two nodes in $I$ are adjacent.”)

• Independent sets have applications to resource optimization, conflict minimization, error-correcting codes, cryptography, and more.
Constraint Optimization with Independent Set and Vertex Cover
What is the *smallest* Independent Set for this graph?
What is the *largest* Vertex Cover for this graph?
What is the *largest* Independent Set for this graph?
What is the *smallest* Vertex Cover for this graph?
A Connection
Independent sets and vertex covers are related.
Independent sets and vertex covers are related.

What’s special about the spiral (¬) nodes?

What’s special about the plus (+) nodes?
Independent sets and vertex covers are related.

What’s special about the spiral (−) nodes?

What’s special about the plus (+) nodes?
Theorem: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. Then $C$ is a vertex cover of $G$ if and only if $V - C$ is an independent set in $G$. 

What’s special about the spiral ($\neg$) nodes?

What’s special about the plus ($+$) nodes?
**Theorem:** Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. Then $C$ is a vertex cover of $G$ if and only if $V - C$ is an independent set in $G$. 
**Theorem:** Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. Then $C$ is a vertex cover of $G$ if and only if $V - C$ is an independent set in $G$. 

How do we prove a **biconditional**? Separately prove the forward and reverse directions of implication.
**Theorem:** Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. Then $C$ is a vertex cover of $G$ if and only if $V - C$ is an independent set in $G$.

**Lemma 1:** Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If $C$ is a vertex cover of $G$, then $V - C$ is an independent set of $G$.

**Lemma 2:** Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If $V - C$ is an independent set of $G$, then $C$ is a vertex cover of $G$. 

**Theorem:** Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. Then $C$ is a vertex cover of $G$ if and only if $V - C$ is an independent set in $G$.

**Lemma 1:** Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If $C$ is a vertex cover of $G$, then $V - C$ is an independent set of $G$.

**Lemma 2:** Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If $V - C$ is an independent set of $G$, then $C$ is a vertex cover of $G$.

**Lemma 2:** Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If $C$ is not a vertex cover of $G$, then $V - C$ is not an independent set in $G$.

It turns out Lemma 2 is easier to prove in its contrapositive form.
**Lemma 1:** Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If $C$ is a vertex cover of $G$, then $V - C$ is an independent set in $G$.

<table>
<thead>
<tr>
<th>What We’re Assuming</th>
<th>What We Need To Show</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G$ is a graph.</td>
<td>$V - C$ is an independent set in $G$.</td>
</tr>
<tr>
<td>$C$ is a vertex cover of $G$.</td>
<td>$\forall x \in V - C. \forall y \in V - C. {x, y} \notin E$.</td>
</tr>
<tr>
<td>$\forall u \in V. \forall v \in V. {u, v} \in E \rightarrow u \in C \vee v \in C$</td>
<td></td>
</tr>
</tbody>
</table>
**Lemma 1:** Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If $C$ is a vertex cover of $G$, then $V - C$ is an independent set in $G$.

<table>
<thead>
<tr>
<th>What We’re Assuming</th>
<th>What We Need To Show</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G$ is a graph.</td>
<td>$V - C$ is an independent set in $G$.</td>
</tr>
<tr>
<td>$C$ is a vertex cover of $G$.</td>
<td>$\forall x \in V - C$.</td>
</tr>
<tr>
<td></td>
<td>$\forall y \in V - C$.</td>
</tr>
<tr>
<td>$\forall u \in V. \forall v \in V. \ ({u, v} \in E \rightarrow$</td>
<td>$\forall y \in V - C$.</td>
</tr>
<tr>
<td></td>
<td>$u \in C \lor v \in C$)</td>
</tr>
</tbody>
</table>

We're assuming a universally-quantified statement. That means we *don’t do anything right now* and instead wait for an edge to present itself. We need to prove a universally-quantified statement. We’ll ask the reader to pick arbitrary choices of $x$ and $y$ for us to work with.
Lemma 1: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If $C$ is a vertex cover of $G$, then $V - C$ is an independent set in $G$.

**What We’re Assuming**

$G$ is a graph.

$C$ is a vertex cover of $G$.

\[
\forall u \in V. \forall v \in V. (\{u, v\} \in E \rightarrow u \in C \lor v \in C)
\]

**What We Need To Show**

$V - C$ is an independent set in $G$.

\[
\forall x \in V - C.
\forall y \in V - C.
\{x, y\} \notin E.
\]

We need to prove a universally-quantified statement. We’ll ask the reader to pick arbitrary choices of $x$ and $y$ for us to work with.
**Lemma 1:** Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If $C$ is a vertex cover of $G$, then $V - C$ is an independent set in $G$.

<table>
<thead>
<tr>
<th>What We’re Assuming</th>
<th>What We Need To Show</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G$ is a graph.</td>
<td>$V - C$ is an independent set in $G$.</td>
</tr>
<tr>
<td>$C$ is a vertex cover of $G$.</td>
<td></td>
</tr>
<tr>
<td>$\forall u \in V. \forall v \in V. \left( {u, v} \in E \rightarrow u \in C \lor v \in C \right)$</td>
<td>$\forall x \in V - C.$</td>
</tr>
<tr>
<td>$x \in V - C.$</td>
<td>$\forall y \in V - C.$</td>
</tr>
<tr>
<td>$y \in V - C.$</td>
<td>${x, y} \notin E.$</td>
</tr>
</tbody>
</table>
**Lemma 1:** Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If $C$ is a vertex cover of $G$, then $V - C$ is an independent set of $G$. 

Proof: Assume $C$ is a vertex cover of $G$. We need to show that $V - C$ is an independent set of $G$. To do so, pick any nodes $x, y \in V - C$; we will show that $\{x, y\} \notin E$.

Suppose for the sake of contradiction that $\{x, y\} \in E$. Because $x, y \in V - C$, we know that $x \notin C$ and $y \notin C$.

However, since $C$ is a vertex cover of $G$ and $\{x, y\} \in E$, we also see that $x \in C$ or $y \in C$, contradicting our previous statement.

We've reached a contradiction, so our assumption was wrong. Therefore, we have $\{x, y\} \notin E$, as required. ■
Lemma 1: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If $C$ is a vertex cover of $G$, then $V - C$ is an independent set of $G$.

Proof:
**Lemma 1:** Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If $C$ is a vertex cover of $G$, then $V - C$ is an independent set of $G$.

**Proof:** Assume $C$ is a vertex cover of $G$. 

There’s no need to introduce $G$ or $C$ here. That’s done in the statement of the lemma itself.
**Lemma 1:** Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If $C$ is a vertex cover of $G$, then $V - C$ is an independent set of $G$.

**Proof:** Assume $C$ is a vertex cover of $G$. We need to show that $V - C$ is an independent set of $G$. 
Lemma 1: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If $C$ is a vertex cover of $G$, then $V - C$ is an independent set of $G$.

Proof: Assume $C$ is a vertex cover of $G$. We need to show that $V - C$ is an independent set of $G$. To do so, pick any nodes $x, y \in V - C$; we will show that $\{x, y\} \notin E$. 

Lemma 1: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If $C$ is a vertex cover of $G$, then $V - C$ is an independent set of $G$.

Proof: Assume $C$ is a vertex cover of $G$. We need to show that $V - C$ is an independent set of $G$. To do so, pick any nodes $x, y \in V - C$; we will show that $\{x, y\} \notin E$.

Suppose for the sake of contradiction that $\{x, y\} \in E$. 
**Lemma 1:** Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If $C$ is a vertex cover of $G$, then $V - C$ is an independent set of $G$.

**Proof:** Assume $C$ is a vertex cover of $G$. We need to show that $V - C$ is an independent set of $G$. To do so, pick any nodes $x, y \in V - C$; we will show that $\{x, y\} \notin E$.

Suppose for the sake of contradiction that $\{x, y\} \in E$. Because $x, y \in V - C$, we know that $x \notin C$ and $y \notin C$. 
Lemma 1: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If $C$ is a vertex cover of $G$, then $V - C$ is an independent set of $G$.

Proof: Assume $C$ is a vertex cover of $G$. We need to show that $V - C$ is an independent set of $G$. To do so, pick any nodes $x, y \in V - C$; we will show that $\{x, y\} \notin E$.

Suppose for the sake of contradiction that $\{x, y\} \in E$. Because $x, y \in V - C$, we know that $x \notin C$ and $y \notin C$. However, since $C$ is a vertex cover of $G$ and $\{x, y\} \in E$, we also see that $x \in C$ or $y \in C$, contradicting our previous statement.
**Lemma 1:** Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If $C$ is a vertex cover of $G$, then $V - C$ is an independent set of $G$.

**Proof:** Assume $C$ is a vertex cover of $G$. We need to show that $V - C$ is an independent set of $G$. To do so, pick any nodes $x, y \in V - C$; we will show that $\{x, y\} \notin E$.

Suppose for the sake of contradiction that $\{x, y\} \in E$. Because $x, y \in V - C$, we know that $x \notin C$ and $y \notin C$. However, since $C$ is a vertex cover of $G$ and $\{x, y\} \in E$, we also see that $x \in C$ or $y \in C$, contradicting our previous statement.

We’ve reached a contradiction, so our assumption was wrong.
**Lemma 1:** Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If $C$ is a vertex cover of $G$, then $V - C$ is an independent set of $G$.

**Proof:** Assume $C$ is a vertex cover of $G$. We need to show that $V - C$ is an independent set of $G$. To do so, pick any nodes $x, y \in V - C$; we will show that $\{x, y\} \notin E$.

Suppose for the sake of contradiction that $\{x, y\} \in E$. Because $x, y \in V - C$, we know that $x \notin C$ and $y \notin C$. However, since $C$ is a vertex cover of $G$ and $\{x, y\} \in E$, we also see that $x \in C$ or $y \in C$, contradicting our previous statement.

We’ve reached a contradiction, so our assumption was wrong. Therefore, we have $\{x, y\} \notin E$, as required.
Lemma 1: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If $C$ is a vertex cover of $G$, then $V - C$ is an independent set of $G$.

Proof: Assume $C$ is a vertex cover of $G$. We need to show that $V - C$ is an independent set of $G$. To do so, pick any nodes $x, y \in V - C$; we will show that $\{x, y\} \notin E$.

Suppose for the sake of contradiction that $\{x, y\} \in E$. Because $x, y \in V - C$, we know that $x \notin C$ and $y \notin C$. However, since $C$ is a vertex cover of $G$ and $\{x, y\} \in E$, we also see that $x \in C$ or $y \in C$, contradicting our previous statement.

We’ve reached a contradiction, so our assumption was wrong. Therefore, we have $\{x, y\} \notin E$, as required. ■
**Theorem:** Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. Then $C$ is a vertex cover of $G$ if and only if $V - C$ is an independent set in $G$.

**Lemma 1:** Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If $C$ is a vertex cover of $G$, then $V - C$ is an independent set of $G$.

**Lemma 2:** Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If $V - C$ is an independent set of $G$, then $C$ is a vertex cover of $G$.

**Lemma 2:** Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If $C$ is not a vertex cover of $G$, then $V - C$ is not an independent set in $G$.

To proceed, we need to take the negations of the FOL definitions of vertex cover and independent set.
Taking Negations

• What is the negation of this statement, which says “$C$ is a vertex cover?”

\[
\forall u \in V. \ \forall v \in V. (\{u, v\} \in E \rightarrow u \in C \lor v \in C)
\]
Taking Negations

• What is the negation of this statement, which says “$C$ is a vertex cover?”

$$
\neg \forall u \in V. \forall v \in V. (\{u, v\} \in E \rightarrow u \in C \lor v \in C)
$$
Taking Negations

• What is the negation of this statement, which says “C is a vertex cover?”

\[ \exists u \in V. \neg \forall v \in V. (\{u, v\} \in E \rightarrow \begin{array}{c} u \in C \lor v \in C \end{array} ) \]
Taking Negations

• What is the negation of this statement, which says “C is a vertex cover?”

\[
\exists u \in V. \exists v \in V. \neg(\{u, v\} \in E \rightarrow u \in C \lor v \in C)
\]
Taking Negations

- What is the negation of this statement, which says "C is a vertex cover?"

\[ \exists u \in V. \exists v \in V. (\{u, v\} \in E \land \neg (u \in C \lor v \in C) ) \]
Taking Negations

- What is the negation of this statement, which says “$C$ is a vertex cover?”

\[
\exists u \in V. \exists v \in V. \ (\{u, v\} \in E \land u \notin C \land v \notin C)
\]
Taking Negations

• What is the negation of this statement, which says “C is a vertex cover?”

\[
\exists u \in V. \exists v \in V. (\{u, v\} \in E \land u \notin C \land v \notin C)
\]

• This says “there is an edge where both endpoints aren’t in C.”
Lemma 2: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If $C$ is not a vertex cover of $G$, then $V - C$ is not an independent set in $G$.

<table>
<thead>
<tr>
<th>What We’re Assuming</th>
<th>What We Need To Show</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G$ is a graph.</td>
<td>$V - C$ is not an ind. set in $G$.</td>
</tr>
<tr>
<td>$C$ is a not a vertex cover of $G.$</td>
<td>$\exists x \in V - C.$</td>
</tr>
<tr>
<td>$\exists u \in V. \exists v \in V. ({u, v} \in E$ $\land$ $u \notin C \land v \notin C$</td>
<td>$\exists y \in V - C.$</td>
</tr>
<tr>
<td></td>
<td>${x, y} \in E.$</td>
</tr>
</tbody>
</table>
**Lemma 2:** Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If $C$ is not a vertex cover of $G$, then $V - C$ is not an independent set in $G$.

<table>
<thead>
<tr>
<th>What We’re Assuming</th>
<th>What We Need To Show</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G$ is a graph.</td>
<td>$V - C$ is not an ind. set in $G$.</td>
</tr>
<tr>
<td>$C$ is a not a vertex cover of $G$.</td>
<td>$\exists x \in V - C$.</td>
</tr>
<tr>
<td>$\exists u \in V. \exists v \in V. ({u, v} \in E \land u \notin C \land v \notin C)$</td>
<td>$\exists y \in V - C$.</td>
</tr>
<tr>
<td></td>
<td>${x, y} \in E$.</td>
</tr>
</tbody>
</table>

We’re assuming an existentially-quantified statement, so we’ll **immediately** introduce variables $u$ and $v$.

We’re proving an existentially-quantified statement, so we **don’t** introduce variables $x$ and $y$. We’re on a scavenger hunt!
**Lemma 2:** Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If $C$ is not a vertex cover of $G$, then $V - C$ is not an independent set in $G$.

<table>
<thead>
<tr>
<th>What We’re Assuming</th>
<th>What We Need To Show</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G$ is a graph.</td>
<td>$V - C$ is not an ind. set in $G$.</td>
</tr>
<tr>
<td>$C$ is a not a vertex cover of $G$.</td>
<td>$\exists x \in V - C$.</td>
</tr>
<tr>
<td></td>
<td>$\exists y \in V - C$.</td>
</tr>
<tr>
<td></td>
<td>${x, y} \in E$.</td>
</tr>
<tr>
<td>$u \in V - C$.</td>
<td></td>
</tr>
<tr>
<td>$v \in V - C$.</td>
<td></td>
</tr>
<tr>
<td>${u, v} \in E$.</td>
<td></td>
</tr>
</tbody>
</table>

We’re assuming an existentially-quantified statement, so we’ll **immediately** introduce variables $u$ and $v$. 
**Lemma 2:** Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If $C$ is not a vertex cover of $G$, then $V - C$ is not an independent set in $G$.

<table>
<thead>
<tr>
<th>What We’re Assuming</th>
<th>What We Need To Show</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G$ is a graph.</td>
<td>$V - C$ is not an ind. set in $G$.</td>
</tr>
<tr>
<td>$C$ is a not a vertex cover of $G$.</td>
<td>$\exists x \in V - C$.</td>
</tr>
<tr>
<td></td>
<td>$\exists y \in V - C$.</td>
</tr>
<tr>
<td></td>
<td>${x, y} \in E$.</td>
</tr>
</tbody>
</table>

Any ideas about what we should pick $x$ and $y$ to be?
**Lemma 2:** Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If $C$ is not a vertex cover of $G$, then $V - C$ is not an independent set of $G$. 
Lemma 2: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If $C$ is not a vertex cover of $G$, then $V - C$ is not an independent set of $G$.

Proof:
Lemma 2: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If $C$ is not a vertex cover of $G$, then $V - C$ is not an independent set of $G$.

Proof: Assume $C$ is not a vertex cover of $G$. 
**Lemma 2:** Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If $C$ is not a vertex cover of $G$, then $V - C$ is not an independent set of $G$.

**Proof:** Assume $C$ is not a vertex cover of $G$. We need to show that $V - C$ is not an independent set of $G$. 
Lemma 2: Let \( G = (V, E) \) be a graph and let \( C \subseteq V \) be a set. If \( C \) is not a vertex cover of \( G \), then \( V - C \) is not an independent set of \( G \).

Proof: Assume \( C \) is not a vertex cover of \( G \). We need to show that \( V - C \) is not an independent set of \( G \).

Since \( C \) is not a vertex cover of \( G \), we know that there exists nodes \( x, y \in V \) where \( \{x, y\} \in E \), where \( x \notin C \), and where \( y \notin C \).
Lemma 2: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If $C$ is not a vertex cover of $G$, then $V - C$ is not an independent set of $G$.

Proof: Assume $C$ is not a vertex cover of $G$. We need to show that $V - C$ is not an independent set of $G$.

Since $C$ is not a vertex cover of $G$, we know that there exists nodes $x, y \in V$ where $\{x, y\} \in E$, where $x \notin C$, and where $y \notin C$. Because $x \in V$ and $x \notin C$, we know that $x \in V - C$. 
**Lemma 2:** Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If $C$ is not a vertex cover of $G$, then $V - C$ is not an independent set of $G$.

**Proof:** Assume $C$ is not a vertex cover of $G$. We need to show that $V - C$ is not an independent set of $G$.

Since $C$ is not a vertex cover of $G$, we know that there exists nodes $x, y \in V$ where $\{x, y\} \in E$, where $x \notin C$, and where $y \notin C$. Because $x \in V$ and $x \notin C$, we know that $x \in V - C$. Similarly, we see that $y \in V - C$.  

\[\]
**Lemma 2:** Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If $C$ is not a vertex cover of $G$, then $V - C$ is not an independent set of $G$.

**Proof:** Assume $C$ is not a vertex cover of $G$. We need to show that $V - C$ is not an independent set of $G$.

Since $C$ is not a vertex cover of $G$, we know that there exists nodes $x, y \in V$ where $\{x, y\} \in E$, where $x \notin C$, and where $y \notin C$. Because $x \in V$ and $x \notin C$, we know that $x \in V - C$. Similarly, we see that $y \in V - C$.

This means that $\{x, y\} \in E$, that $x \in V - C$, and that $y \in V - C$, and therefore that $V - C$ is not an independent set of $G$, as required.
Lemma 2: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If $C$ is not a vertex cover of $G$, then $V - C$ is not an independent set of $G$.

Proof: Assume $C$ is not a vertex cover of $G$. We need to show that $V - C$ is not an independent set of $G$.

Since $C$ is not a vertex cover of $G$, we know that there exists nodes $x, y \in V$ where $\{x, y\} \in E$, where $x \notin C$, and where $y \notin C$. Because $x \in V$ and $x \notin C$, we know that $x \in V - C$. Similarly, we see that $y \in V - C$.

This means that $\{x, y\} \in E$, that $x \in V - C$, and that $y \in V - C$, and therefore that $V - C$ is not an independent set of $G$, as required. ■
Recap for Today

- A **graph** is a structure for representing items that may be linked together. **Digraphs** represent that same idea, but with a directionality on the links.

- Graphs can’t have **self-loops**; digraphs can.

- **Vertex covers** and **independent sets** are useful tools for modeling problems with graphs.

- The complement of a vertex cover is an independent set, and vice-versa.
Next Time

- **Paths and Trails**
  - Walking from one point to another.

- **Indegrees and Outdegrees**
  - Counting how many neighbors you have, in the directed case.

- **Teleporting a Train**
  - Can you get stuck in a loop?

- **The Cantor-Bernstein-Schroeder Theorem**
  - A proof on set cardinality that’s really a proof about graphs.