Graph Theory
Part One
Graph Theory

For those of you who have already completed CS106B/X:
Chemical Bonds
What's in Common

- Each of these structures consists of
  - a collection of objects and
  - links between those objects.
- **Goal:** find a general framework for describing these objects and their properties.
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A **graph** is a mathematical structure for representing relationships.

A graph consists of a set of **nodes** (or **vertices**) connected by **edges** (or **arcs**).
Some graphs are *directed*. On these sites, you can follow someone who doesn’t follow you back.

A tournament diagram shows who beat who.
Some graphs are *undirected*.

On this site, if you are friends with someone, they are also friends with you.

Atoms that are adjacent to each other in a molecule.

Words that differ from each other by exactly one letter.
Going forward, we're exclusively focused on undirected graphs.

The term “graph” will mean undirected graphs with a finite number of nodes, (unless specified otherwise).
Formalizing Graphs

• How might we define a graph mathematically?

• We need to specify
  • what the nodes in the graph are, and
  • which edges are in the graph.

• The nodes can be pretty much anything.

• What about the edges?
Formalizing Graphs

- An **unordered pair** is a set \( \{a, b\} \) of two elements \( a \neq b \). (Remember that sets are unordered).
  - \( \{0, 1\} = \{1, 0\} \)
- An **undirected graph** is an ordered pair \( G = (V, E) \), where
  - \( V \) is a set of nodes, which can be anything, and
  - \( E \) is a set of edges, which are unordered pairs of nodes drawn from \( V \).
- **[For your reference, but remember we won’t be focusing on them in this class]** A **directed graph** is an ordered pair \( G = (V, E) \), where
  - \( V \) is a set of nodes, which can be anything, and
  - \( E \) is a set of edges, which are *ordered* pairs of nodes drawn from \( V \).
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How many of these drawings are of valid undirected graphs?

Answer at PollEv.com/cs103 or text CS103 to 22333 once to join, then a number.
Self-Loops

- An edge from a node to itself is called a **self-loop**.
- In undirected graphs, self-loops are generally not allowed.
  - Can you see how this follows from the definition?
- In directed graphs, self-loops are generally allowed unless specified otherwise.
Standard Graph Terminology
Two nodes are called *adjacent* if there is an edge between them.
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Adjacent nodes

- Let $G = (V, E)$ be a graph.
  - Intuitively, two nodes are adjacent if they're linked by an edge.
- Formally speaking, we say that two nodes $u, v \in V$ are adjacent if $\{u, v\} \in E$. 

To

From

SF, Sac, LA, Phoe, Flag, Bar, LV, Mon, SLC, But, Sea
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(This path has length 10, but visits 11 cities.)
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How many paths in this graph are there from SF to LA?

A. 1
B. 4
C. 10
D. 20
E. None of these.
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(A cycle, not a simple cycle.)
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(These nodes are not connected. No Grand Canyon for you.)
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