Graph Theory
Part One
Graph Theory

For those of you who have already completed CS106B/X:
Chemical Bonds
Zombies are everywhere

Is this guy a zombie?

Yes: Can you escape?

Yes: Hide in a small confined space with a group of people from different backgrounds

No: Yes he is

Death.
What's in Common

• Each of these structures consists of
  • a collection of objects and
  • links between those objects.

• **Goal:** find a general framework for describing these objects and their properties.
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A graph consists of a set of **nodes** (or **vertices**) connected by **edges** (or **arcs**).
Some graphs are *directed*. 
Some graphs are *undirected*.
Going forward, we're primarily going to focus on undirected graphs.

The term “graph” generally refers to undirected graphs with a finite number of nodes, unless specified otherwise.
Formalizing Graphs

• How might we define a graph mathematically?

• We need to specify
  • what the nodes in the graph are, and
  • which edges are in the graph.

• The nodes can be pretty much anything.

• What about the edges?
Formalizing Graphs

• An **unordered pair** is a set \( \{a, b\} \) of two elements \( a \neq b \). (Remember that sets are unordered).
  • \( \{0, 1\} = \{1, 0\} \)

• An **undirected graph** is an ordered pair \( G = (V, E) \), where
  • \( V \) is a set of nodes, which can be anything, and
  • \( E \) is a set of edges, which are unordered pairs of nodes drawn from \( V \).

• [For your reference, but remember we won’t be focusing on them in this class] A **directed graph** is an ordered pair \( G = (V, E) \), where
  • \( V \) is a set of nodes, which can be anything, and
  • \( E \) is a set of edges, which are **ordered** pairs of nodes drawn from \( V \).
- An **unordered pair** is a set \( \{a, b\} \) of two elements \( a \neq b \).
- An **undirected graph** is an ordered pair \( G = (V, E) \), where
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How many of these drawings are of valid undirected graphs?

Answer at PollEv.com/cs103 or text **CS103** to **22333** once to join, then a number.
Self-Loops

- An edge from a node to itself is called a **self-loop**.
- In undirected graphs, self-loops are generally not allowed.
  - Can you see how this follows from the definition?
- In directed graphs, self-loops are generally allowed unless specified otherwise.
Standard Graph Terminology
Two nodes are called *adjacent* if there is an edge between them.
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Using our Formalisms

- Let $G = (V, E)$ be a graph.
- Intuitively, two nodes are adjacent if they're linked by an edge.
- Formally speaking, we say that two nodes $u, v \in V$ are **adjacent** if $\{u, v\} \in E$. 
SF, Sac, Port, Sea
From SF, Sac, SLC, Port, Sea
To SF, Sac, SLC, Port, Sea
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The length of the path $v_1, \ldots, v_n$ is $n - 1$. 

SF, Sac, LA, Phoe, Flag, Bar, LV, Mon, SLC, But, Sea
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(This path has length 10, but visits 11 cities.)
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(This cycle has length nine and visits nine different cities.)

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How many paths in this graph are there from SF to LA?

A. 1  
B. 4  
C. 10  
D. 20  
E. None of these.
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A simple path in a graph is path that does not repeat any nodes or edges.

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SF, Sac, LA, Phoe, Flag, Bar, LA

(This path has length six.)
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Sac, SLC, Port, Sac, SLC
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Two nodes in a graph are called \textit{connected} if there is a path between them.
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(These nodes are not connected. No Grand Canyon for you.)
A path in a graph $G = (V, E)$ is a sequence of one or more nodes $v_1, v_2, v_3, \ldots, v_n$ such that any two consecutive nodes in the sequence are adjacent.

Two nodes in a graph are called **connected** if there is a path between them.

A graph $G$ as a whole is called **connected** if all pairs of nodes in $G$ are connected.
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A graph $G$ as a whole is called connected if all pairs of nodes in $G$ are connected.

(This graph is not connected.)