In class, I alluded to some other things that Turing machines can do such as working with base 10 numbers and doing simple arithmetic operations like addition.

Here’s a few more TM constructions to show how to go and do that! Going through these slides is completely optional, but we thought we’d include them in case you’re curious :)}
TM Arithmetic

• Let's design a TM that, given a tape that looks like this:

  ... 1 3 7 4 2 ...

  ends up having the tape look like this:

  ... 1 7 9 0 0 ...

• In other words, we want to build a TM that can add two numbers.
TM Arithmetic

• There are many ways we could in principle design this TM.

• We're going to take the following approach:
  - First, we'll build a TM that increments a number.
  - Next, we'll build a TM that decrements a number.
  - Then, we'll combine them together, repeatedly decrementing the second number and adding one to the first number.
TM Arithmetic

add(num1, num2) {
    while (num2 > 0) {
        increment(num1);
        decrement(num2);
    }
}

TM Arithmetic

```c
add(num1, num2) {
    while (num2 > 0) {
        increment(num1);
        decrement(num2);
    }
}
```

Let's write this helper method first!
Incrementing Numbers

- Let's begin by building a TM that increments a number.
- We'll assume that
  - the tape head points at the start of a number,
  - there are at least two blanks to the left of the number, and
  - that there's at least one blank at the end of the number.
- The tape head will end at the start of the number after incrementing it.
Incrementing Numbers

- Let's begin by building a TM that increments a number.
- We'll assume that
  - the tape head points at the start of a number,
  - there is at least two blanks to the left of the number, and
  - that there's at least one blank at the end of the number.
- The tape head will end at the start of the number after incrementing it.
Incrementing Numbers

- Let's begin by building a TM that increments a number.
- We'll assume that
  - the tape head points at the start of a number,
  - there is at least two blanks to the left of the number, and
  - there's at least one blank at the end of the number.
- The tape head will end at the start of the number after incrementing it.
Incrementing Numbers

- Let's begin by building a TM that increments a number.

- We'll assume that
  - the tape head points at the start of a number,
  - there is at least two blanks to the left of the number, and
  - that there's at least one blank at the end of the number.

- The tape head will end at the start of the number after incrementing it.
Incrementing Numbers

• Let's begin by building a TM that increments a number.

• We'll assume that
  – the tape head points at the start of a number,
  – there are at least two blanks to the left of the number, and
  – that there's at least one blank at the end of the number.

• The tape head will end at the start of the number after incrementing it.
Incrementing Numbers

- Let's begin by building a TM that increments a number.
- We'll assume that
  - the tape head points at the start of a number,
  - there are at least two blanks to the left of the number, and
  - there's at least one blank at the end of the number.
- The tape head will end at the start of the number after incrementing it.
Incrementing Numbers

• Let's begin by building a TM that increments a number.

• We'll assume that
  – the tape head points at the start of a number,
  – there is one blank to the left of the number, and
  – that there's at least one blank at the end of the number.

• The tape head will end at the start of the number after incrementing it.
Incrementing Numbers

- Let's begin by building a TM that increments a number.
- We'll assume that
  - the tape head points at the start of a number,
  - there are at least two blanks to the left of the number, and
  - that there's at least one blank at the end of the number.
- The tape head will end at the start of the number after incrementing it.
Incrementing Numbers

- Let's begin by building a TM that increments a number.
- We'll assume that
  - the tape head points at the start of a number,
  - there is at least two blanks to the left of the number, and
  - that there's at least one blank at the end of the number.
- The tape head will end at the start of the number after incrementing it.

... 9 9 9
Incrementing Numbers

• Let's begin by building a TM that increments a number.

• We'll assume that
  - the tape head points at the start of a number,
  - there is at least two blanks to the left of the number, and
  - that there's at least one blank at the end of the number.

• The tape head will end at the start of the number after incrementing it.

... 9 9 9
Incrementing Numbers

• Let's begin by building a TM that increments a number.

• We'll assume that
  - the tape head points at the start of a number,
  - there is at least two blanks to the left of the number, and
  - that there's at least one blank at the end of the number.

• The tape head will end at the start of the number after incrementing it.
Incrementing Numbers

• Let's begin by building a TM that increments a number.

• We'll assume that
  – the tape head points at the start of a number,
  – there is at least two blanks to the left of the number, and
  – that there's at least one blank at the end of the number.

• The tape head will end at the start of the number after incrementing it.
Incrementing Numbers

- Let's begin by building a TM that increments a number.
- We'll assume that
  - the tape head points at the start of a number,
  - there is at least two blanks to the left of the number, and
  - that there's at least one blank at the end of the number.
- The tape head will end at the start of the number after incrementing it.
Incrementing Numbers

• Let's begin by building a TM that increments a number.

• We'll assume that
  – the tape head points at the start of a number,
  – there is at least two blanks to the left of the number, and
  – that there's at least one blank at the end of the number.

• The tape head will end at the start of the number after incrementing it.
Let's begin by building a TM that increments a number.

We'll assume that
- the tape head points at the start of a number,
- there is at least two blanks to the left of the number, and
- that there's at least one blank at the end of the number.

The tape head will end at the start of the number after incrementing it.
Incrementing Numbers

- Let's begin by building a TM that increments a number.
- We'll assume that
  - the tape head points at the start of a number,
  - there is at least two blanks to the left of the number, and
  - that there's at least one blank at the end of the number.
- The tape head will end at the start of the number after incrementing it.
Incrementing Numbers

• Let's begin by building a TM that increments a number.
• We'll assume that
  – the tape head points at the start of a number,
  – there is at least two blanks to the left of the number, and
  – that there's at least one blank at the end of the number.
• The tape head will end at the start of the number after incrementing it.

... 1 0 0 0
Incrementing Numbers

• Let's begin by building a TM that increments a number.
• We'll assume that
  – the tape head points at the start of a number,
  – there is at least two blanks to the left of the number, and
  – that there's at least one blank at the end of the number.
• The tape head will end at the start of the number after incrementing it.
Incrementing Numbers

```plaintext
increment(num) {
    go to the end of the number;
    while (the current digit is 9) {
        set the current digit to 0;
        back up one digit;
    }
    increment the current digit;
    go to the start of the number;
}
```
<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>9</th>
<th>9</th>
<th></th>
</tr>
</thead>
</table>

... → 0, R
1 → 1, R
... → End
9 → 9, R
0 → 0, R
1 → 1, R
...
9 → 9, R
0 → 0, R
1 → 1, R
...
9 → 9, R

start
To
End
0 → 0, R
1 → 1, R
...
9 → 9, R
0 → 0, R
1 → 1, R
...
9 → 9, R
To End

0 → 0, R
1 → 1, R
...
9 → 9, R

□ → □, L

start
0 → 0, R
1 → 1, R
... 
9 → 9, R

Start

To
End

☐ → ☐, L

Wrap
Nines

9 → 0, L

... 1 2 9 9 ...

...
0 → 0, R
1 → 1, R
... 
9 → 9, R

start

To End

□ → □, L

Wrap Nines

9 → 0, L
start

\[0 \rightarrow 0, \text{R}\]
\[1 \rightarrow 1, \text{R}\]
\[\ldots\]
\[9 \rightarrow 9, \text{R}\]

To

End

\[\square \rightarrow \square, \text{L}\]

Wrap

Nines

\[9 \rightarrow 0, \text{L}\]

\[0 \rightarrow 1, \text{L}\]
\[1 \rightarrow 2, \text{L}\]
\[2 \rightarrow 3, \text{L}\]
\[\ldots\]
\[8 \rightarrow 9, \text{L}\]

\[0 \rightarrow 0, \text{L}\]
\[1 \rightarrow 1, \text{L}\]
\[\ldots\]
\[9 \rightarrow 9, \text{L}\]

Back

Home

… | 1 | 3 | 0 | 0 | …
start

0 → 0, R
1 → 1, R
...
9 → 9, R

To End

□ → □, L

Wrap Nines

9 → 0, L

0 → 1, L
1 → 2, L
2 → 3, L
...
8 → 9, L

□ → □, R

Back Home

0 → 0, L
1 → 1, L
...
9 → 9, L

... 1 3 0 0 ...
0 → 0, R
1 → 1, R
... 9 → 9, R

To End

□ → □, L

Wrap Nines

9 → 0, L

0 → 1, L
1 → 2, L
2 → 3, L
... 8 → 9, L

done!

□ → □, R

Back Home

0 → 0, L
1 → 1, L
... 9 → 9, L

... 1 3 0 0 ...

...
0 → 0, R
1 → 1, R
... 9 → 0, R

0 → 0, L
1 → 1, L
... 8 → 9, L

0 → 0, L
1 → 1, L
... 9 → 9, L

... 1 3 0 0 ...
0 → 0, R
1 → 1, R
...
9 → 9, R

\[\text{To End} \rightarrow \text{Wrap Nines}\]

\[\square \rightarrow \square, L\]

9 → 0, L

0 → 1, L
1 → 2, L
2 → 3, L
...
8 → 9, L

\[\text{Wrap Nines} \rightarrow \text{Back Home}\]

0 → 0, L
1 → 1, L
...
9 → 9, L

\[\square \rightarrow \square, R\]

\[\text{Back Home} \rightarrow \text{done!}\]
To End

0 → 0, R
1 → 1, R
...
9 → 9, R

Wrap Nines

9 → 0, L
0 → 1, L
1 → 2, L
2 → 3, L
...
8 → 9, L

done!

Back Home

0 → 0, L
1 → 1, L
...
9 → 9, L

... 1 0 0 2 ...

...
0 → 0, R
1 → 1, R
...
9 → 9, R

□ → □, L

0 → 1, L
1 → 2, L
2 → 3, L
...
8 → 9, L

□ → □, R

0 → 0, L
1 → 1, L
...
9 → 9, L

done!

Start

To End

Wrap Nines

Back Home

Home
0 → 0, R
1 → 1, R
...
9 → 9, R

□ → □, L

Wrap Nines

9 → 0, L
0 → 1, L
1 → 2, L
2 → 3, L
...
8 → 9, L

Back Home

0 → 0, L
1 → 1, L
...
9 → 9, L

done!

□ → □, R

To End
0 → 0, R
1 → 1, R
...
9 → 9, R

0 → 1, L
1 → 2, L
2 → 3, L
...
8 → 9, L

0 → 0, L
1 → 1, L
...
9 → 9, L

done!

... 1 0 0 2 ...
...
start

0 → 0, R
1 → 1, R
...
9 → 9, R

To End

□ → □, L

Wrap Nines

9 → 0, L

0 → 1, L
1 → 2, L
2 → 3, L
...
8 → 9, L

done!

□ → □, R

Back Home

0 → 0, L
1 → 1, L
...
9 → 9, L

... 1 0 0 2 ...

...
0 → 0, R
1 → 1, R
... 9 → 9, R

To End

Done!

□ → □, R

Back Home

Wrap Nines

9 → 0, L
0 → 1, L
1 → 2, L
2 → 3, L
... 8 → 9, L

0 → 0, L
1 → 1, L
... 9 → 9, L

... 1 0 0 3 ...
1 → 0, R
1 → 1, R
...
9 → 9, R

start

 To End

□ → □, L

Wrap Nines

9 → 0, L
0 → 1, L
1 → 2, L
2 → 3, L
...
8 → 9, L

done!

□ → □, R

Back Home

0 → 0, L
1 → 1, L
...
9 → 9, L

… 1 0 0 3 …
\[ \begin{align*}
0 & \rightarrow 0, R \\
1 & \rightarrow 1, R \\
\ldots & \\
9 & \rightarrow 9, R \\
\rightarrow & \\
0 & \rightarrow 0, L \\
1 & \rightarrow 1, L \\
2 & \rightarrow 2, L \\
\ldots & \\
8 & \rightarrow 9, L \\
\rightarrow & \\
0 & \rightarrow 0, L \\
1 & \rightarrow 1, L \\
\ldots & \\
9 & \rightarrow 9, L \\
\rightarrow & \\
\end{align*} \]
start

0 → 0, R
1 → 1, R
...
9 → 9, R

To End

□ → □, L

Wrap Nines

9 → 0, L
0 → 1, L
1 → 2, L
2 → 3, L
...
8 → 9, L

Back Home

□ → □, R

done!

0 → 0, L
1 → 1, L
...
9 → 9, L
To End

0 → 0, R
1 → 1, R
...
9 → 9, R

Wrap Nines

9 → 0, L
0 → 1, L
1 → 2, L
2 → 3, L
...
8 → 9, L
9 → 9, L

Back Home

0 → 0, L
1 → 1, L
...
9 → 9, L

done!

... 9 9 9 ...

start
To End

0 → 0, R
1 → 1, R
...
9 → 9, R

Wrap Nines

9 → 0, L
0 → 1, L
1 → 2, L
2 → 3, L
...
8 → 9, L

Back Home

0 → 0, L
1 → 1, L
...
9 → 9, L

done!

Square → Square, R

... 9 9 9 ...
0 → 0, R
1 → 1, R
... 9 → 9, R

To End

□ → □, L

Wrap Nines

9 → 0, L

0 → 1, L
1 → 2, L
2 → 3, L
... 8 → 9, L

done!

□ → □, R

Back Home

0 → 0, L
1 → 1, L
... 9 → 9, L

... 9 9 9 ...

...
0 → 0, R
1 → 1, R
...
9 → 9, R

\[ \square \rightarrow \square, L \]

To End

Wrap Nines

9 → 0, L
0 → 1, L
1 → 2, L
2 → 3, L
...
8 → 9, L

\[ \square \rightarrow \square, R \]

done!

Back Home

0 → 0, L
1 → 1, L
...
9 → 9, L

... 9 9 9 ...

...
To End

0 → 0, R
1 → 1, R
...
9 → 9, R

Wrap Nines

9 → 0, L
0 → 1, L
1 → 2, L
2 → 3, L
...
8 → 9, L

Back Home

0 → 0, L
1 → 1, L
...
9 → 9, L

done!

start

... 9 0 0 ...
To End

0 → 0, R
1 → 1, R
...
9 → 9, R

Wrap Nines

9 → 0, L

Back Home

0 → 0, L
1 → 1, L
...
9 → 9, L

done!

0 → 0, R
1 → 1, R
...
9 → 9, L

start

0 0 0
\[
\begin{align*}
0 & \rightarrow 0, \text{ R} \\
1 & \rightarrow 1, \text{ R} \\
\cdots \\
9 & \rightarrow 9, \text{ R} \\
\square & \rightarrow \square, \text{ L} \\
9 & \rightarrow 0, \text{ L} \\
0 & \rightarrow 1, \text{ L} \\
1 & \rightarrow 2, \text{ L} \\
2 & \rightarrow 3, \text{ L} \\
\cdots \\
8 & \rightarrow 9, \text{ L} \\
\square & \rightarrow 1, \text{ L} \\
0 & \rightarrow 0, \text{ L} \\
1 & \rightarrow 1, \text{ L} \\
\cdots \\
9 & \rightarrow 9, \text{ L} \\
\end{align*}
\]
Decrementing Numbers

• Now, let's build a TM that decrements a number.
• We'll assume that
  – the tape head points at the start of a number,
  – there is at least one blank on each side of the number.
• The tape head will end at the start of the number after decrementing it.
• If the number is 0, then the subroutine should somehow signal this rather than making the number negative.
Decrementing Numbers

- Now, let's build a TM that decrements a number.
- We'll assume that
  - the tape head points at the start of a number,
  - there is at least one blank on each side of the number.
- The tape head will end at the start of the number after decrementing it.
- If the number is 0, then the subroutine should somehow signal this rather than making the number negative.
Decrementing Numbers

- Now, let's build a TM that decrements a number.
- We'll assume that
  - the tape head points at the start of a number,
  - there is at least one blank on each side of the number.
- The tape head will end at the start of the number after decrementing it.
- If the number is 0, then the subroutine should somehow signal this rather than making the number negative.
Decrementing Numbers

Now, let's build a TM that decrements a number.

We'll assume that
- the tape head points at the start of a number,
- there is at least one blank on each side of the number.

The tape head will end at the start of the number after decrementing it.

If the number is 0, then the subroutine should somehow signal this rather than making the number negative.
Decrementing Numbers

• Now, let's build a TM that decrements a number.
• We'll assume that
  – the tape head points at the start of a number,
  – there is at least one blank on each side of the number.
• The tape head will end at the start of the number after decrementing it.
• If the number is 0, then the subroutine should somehow signal this rather than making the number negative.
Decrementing Numbers

• Now, let's build a TM that decrements a number.
• We'll assume that
  – the tape head points at the start of a number,
  – there is at least one blank on each side of the number.
• The tape head will end at the start of the number after decrementing it.
• If the number is 0, then the subroutine should somehow signal this rather than making the number negative.
Decrementing Numbers

- Now, let's build a TM that decrements a number.
- We'll assume that
  - the tape head points at the start of a number,
  - there is at least one blank on each side of the number.
- The tape head will end at the start of the number after decrementing it.
- If the number is 0, then the subroutine should somehow signal this rather than making the number negative.
Decrementing Numbers

- Now, let's build a TM that decrements a number.
- We'll assume that
  - the tape head points at the start of a number,
  - there is at least one blank on each side of the number.
- The tape head will end at the start of the number after decrementing it.
- If the number is 0, then the subroutine should somehow signal this rather than making the number negative.
Decrementing Numbers

• Now, let's build a TM that decrements a number.
• We'll assume that
  - the tape head points at the start of a number,
  - there is at least one blank on each side of the number.
• The tape head will end at the start of the number after decrementing it.
• If the number is 0, then the subroutine should somehow signal this rather than making the number negative.
Decrementing Numbers

• Now, let's build a TM that decrements a number.
• We'll assume that
  – the tape head points at the start of a number,
  – there is at least one blank on each side of the number.
• The tape head will end at the start of the number after decrementing it.
• If the number is 0, then the subroutine should somehow signal this rather than making the number negative.
Decrementing Numbers

- Now, let's build a TM that decrements a number.
- We'll assume that
  - the tape head points at the start of a number,
  - there is at least one blank on each side of the number.
- The tape head will end at the start of the number after decrementing it.
- If the number is 0, then the subroutine should somehow signal this rather than making the number negative.
Decrementing Numbers

• Now, let's build a TM that decrements a number.
• We'll assume that
  – the tape head points at the start of a number,
  – there is at least one blank on each side of the number.
• The tape head will end at the start of the number after decrementing it.
• If the number is 0, then the subroutine should somehow signal this rather than making the number negative.

1 0 1
Decrementing Numbers

• Now, let's build a TM that decrements a number.
• We'll assume that
  – the tape head points at the start of a number,
  – there is at least one blank on each side of the number.
• The tape head will end at the start of the number after decrementing it.
• If the number is 0, then the subroutine should somehow signal this rather than making the number negative.
Decrementing Numbers

• Now, let's build a TM that decrements a number.
• We'll assume that
  – the tape head points at the start of a number,
  – there is at least one blank on each side of the number.
• The tape head will end at the start of the number after decrementing it.
• If the number is 0, then the subroutine should somehow signal this rather than making the number negative.
Decrementing Numbers

• Now, let's build a TM that decrements a number.
• We'll assume that
  – the tape head points at the start of a number,
  – there is at least one blank on each side of the number.
• The tape head will end at the start of the number after decrementing it.
• If the number is 0, then the subroutine should somehow signal this rather than making the number negative.
Decrementing Numbers

- Now, let's build a TM that decrements a number.
- We'll assume that
  - the tape head points at the start of a number,
  - there is at least one blank on each side of the number.
- The tape head will end at the start of the number after decrementing it.
- If the number is 0, then the subroutine should somehow signal this rather than making the number negative.
Decrementing Numbers

- Now, let's build a TM that decrements a number.
- We'll assume that
  - the tape head points at the start of a number,
  - there is at least one blank on each side of the number.
- The tape head will end at the start of the number after decrementing it.
- If the number is 0, then the subroutine should somehow signal this rather than making the number negative.
Decrementing Numbers

• Now, let's build a TM that decrements a number.
• We'll assume that
  – the tape head points at the start of a number,
  – there is at least one blank on each side of the number.
• The tape head will end at the start of the number after decrementing it.
• If the number is 0, then the subroutine should somehow signal this rather than making the number negative.
Decrementing Numbers

• Now, let's build a TM that decrements a number.
• We'll assume that
  – the tape head points at the start of a number,
  – there is at least one blank on each side of the number.
• The tape head will end at the start of the number after decrementing it.
• If the number is 0, then the subroutine should somehow signal this rather than making the number negative.

1 0 0
Decrementing Numbers

• Now, let's build a TM that decrements a number.
• We'll assume that
  – the tape head points at the start of a number,
  – there is at least one blank on each side of the number.
• The tape head will end at the start of the number after decrementing it.
• If the number is 0, then the subroutine should somehow signal this rather than making the number negative.
Decrementing Numbers

- Now, let's build a TM that decrements a number.
- We'll assume that
  - the tape head points at the start of a number,
  - there is at least one blank on each side of the number.
- The tape head will end at the start of the number after decrementing it.
- If the number is 0, then the subroutine should somehow signal this rather than making the number negative.
Decrementing Numbers

• Now, let's build a TM that decrements a number.
• We'll assume that
  – the tape head points at the start of a number,
  – there is at least one blank on each side of the number.
• The tape head will end at the start of the number after decrementing it.
• If the number is 0, then the subroutine should somehow signal this rather than making the number negative.
Decrementing Numbers

- Now, let's build a TM that decrements a number.
- We'll assume that
  - the tape head points at the start of a number,
  - there is at least one blank on each side of the number.
- The tape head will end at the start of the number after decrementing it.
- If the number is 0, then the subroutine should somehow signal this rather than making the number negative.

![Image of a tape with numbers]
Decrementing Numbers

• Now, let's build a TM that decrements a number.
• We'll assume that
  – the tape head points at the start of a number,
  – there is at least one blank on each side of the number.
• The tape head will end at the start of the number after decrementing it.
• If the number is 0, then the subroutine should somehow signal this rather than making the number negative.
Decrementing Numbers

- Now, let's build a TM that decrements a number.
- We'll assume that
  - the tape head points at the start of a number,
  - there is at least one blank on each side of the number.
- The tape head will end at the start of the number after decrementing it.
- If the number is 0, then the subroutine should somehow signal this rather than making the number negative.
Decrementing Numbers

decrement(num) {
    go to the end of the number;
    if (every digit was 0) {
        signal that we're done;
    }
    while (the current digit is 0) {
        set the current digit to 9;
        back up one digit;
    }
    decrement the current digit;
    go to the start of the number;
}
start

... 0 2 0 0 ...

...
Non-zero?
0 → 0, R
Non-zero?

0 → 0, R
1 → 1, R
2 → 2, R
...
9 → 9, R

start

... 0 2 0 0 ...
To End

1 → 1, R
2 → 2, R
... 9 → 9, R
0 → 0, R

Non-zero?

... 0 2 0 0 ...
To End

1 → 1, R
2 → 2, R
...
9 → 9, R

0 → 0, R

Non-zero?

0 2 0 0
0 → 0, R
1 → 1, R
2 → 2, R
...
9 → 9, R

Non-zero?

0 → 0, R

To End

start

... 0 2 0 0 ...

...
0 → 0, R
1 → 1, R
... 9 → 9, R

Non-zero?

0 → 0, R
1 → 1, R
2 → 2, R
... 9 → 9, R

Start

To End

□ → □, L

Wrap Zeros

0 → 9, L

... 0 2 9 9...

...
To End

Non-zero?

Wrap Zeros

0 → 0, R
1 → 1, R
...
9 → 9, R

0 → 0, R
1 → 1, R
2 → 2, R
...
9 → 9, R

start

0 → 0, R
1 → 0, L
2 → 1, L
3 → 2, L
...
9 → 8, L

... 0 1 9 9 ...
Non-zero?

To End

Wrap Zeros

0 → 0, R
1 → 1, R
...
9 → 9, R

1 → 1, R
2 → 2, R
...
9 → 9, R

start

Non-zero?

0 → 0, R

□ → □, L

Back Home

done!

□ → □, R

1 → 0, L
2 → 1, L
3 → 2, L
...
9 → 8, L

0 → 0, L
1 → 1, L
...
9 → 9, L

... 0 1 9 9 ...
To End Wrap Zeros

Non-zero?

start

0 → 0, R
1 → 1, R
...
9 → 9, R

0 → 0, R
1 → 1, R
2 → 2, R
...
9 → 9, R

do!

0 → 0, L
1 → 0, L
2 → 1, L
3 → 2, L
...
9 → 8, L

Back Home

0 → 0, L
1 → 1, L
...
9 → 9, L

0 1 9 9
0 → 0, R
1 → 1, R
...
9 → 9, R

0 → 0, R
1 → 1, R
2 → 2, R
...
9 → 9, R

\text{start}

\text{To End}

\square \rightarrow \square, L

0 → 9, L
1 → 0, L
2 → 1, L
3 → 2, L
...
9 → 8, L

\text{Wrap Zeros}

\text{To End}

\text{Non-zero?}

\text{Back Home}

\text{done!}

\text{Back Home}

\text{Home}
Non-zero?

To End

0 → 0, R
1 → 1, R
...
9 → 9, R

Wrap Zeros

0 → 9, L
1 → 0, L
2 → 1, L
3 → 2, L
...
9 → 8, L

Back Home

done!

0 → 0, L
1 → 1, L
...
9 → 9, L

start

Non-zero?

0 → 0, R
1 → 1, R
2 → 2, R
...
9 → 9, R

□ → □, L

□ → □, R
... | 0 | 0 | 0 | 0 | 0 | ...

start

Non-zero?

To End

\[
\begin{align*}
0 &\rightarrow 0, R \\
1 &\rightarrow 1, R \\
2 &\rightarrow 2, R \\
\vdots &\rightarrow \square, \square, L \\
9 &\rightarrow 9, R
\end{align*}
\]

Wrap Zeros

\[
\begin{align*}
0 &\rightarrow 9, L \\
1 &\rightarrow 0, L \\
2 &\rightarrow 1, L \\
3 &\rightarrow 2, L \\
\vdots &\rightarrow \square, \square, R \\
9 &\rightarrow 8, L
\end{align*}
\]

Back Home

\[
\begin{align*}
0 &\rightarrow 0, L \\
1 &\rightarrow 1, L \\
\vdots &\rightarrow \square, \square, L \\
9 &\rightarrow 9, L
\end{align*}
\]

done!
Non-zero?

To End

Wrap Zeros

done!

Non-zero?

start

Back Home

... 0 → 0, R
1 → 1, R
... 9 → 9, R

0 → 0, R
1 → 1, R
2 → 2, R
... 9 → 9, R

0 → 0, R
1 → 1, L
2 → 2, L
... 9 → 8, L

1 → 0, L
2 → 1, L
3 → 2, L
... 9 → 9, L

0 → 0, L
1 → 1, L
... 9 → 9, L

... 0 0 0 0 0
In this diagram, we have a sequence of states and transitions. The states include:

- **Non-zero?**
- **To End**
- **Wrap Zeros**
- **Back Home**

The transitions are as follows:

- From **Non-zero?**:
  - **0 → 0, R**
  - **1 → 1, R**
  - **2 → 2, R**
  - **...**
  - **9 → 9, R**

- From **To End**:
  - **□ → □, L**

- From **Wrap Zeros**:
  - **0 → 9, L**

- From **Back Home**:
  - **□ → □, R**

The input sequence at the bottom of the diagram is:

```
... 0 0 0 0 0 ...
```
TM Subroutines

• Sometimes, a subroutine needs to report back some information about what happened.

• Just as a function can return multiple different values, we'll allow subroutines to have different “done” states.

• Each state can then be wired to a different state, so a TM using the subroutine can control what happens next.
Putting it All Together

• Our goal is to build a TM that, given two numbers, adds those numbers together.

• Before:

... | 1 | 3 | 7 | 4 | 2 | ...

• After:

... | 1 | 7 | 9 | 0 | 0 | ...
add(num1, num2) {
    while (num2 > 0) {
        increment(num1);
        decrement(num2);
    }
}

TM Arithmetic
Using Our Subroutines

- We'll build our new machine using our existing increment and decrement subroutines:
Using Our Subroutines

- We'll build our new machine using our existing increment and decrement subroutines:
Using Our Subroutines

- We'll build our new machine using our existing increment and decrement subroutines:

... 1 3 7 4 2 ...
Using Our Subroutines

- We'll build our new machine using our existing increment and decrement subroutines:
Using Our Subroutines

• We'll build our new machine using our existing increment and decrement subroutines:
Using Our Subroutines

• We'll build our new machine using our existing increment and decrement subroutines:
Using Our Subroutines

- We'll build our new machine using our existing increment and decrement subroutines:
Using Our Subroutines

• We'll build our new machine using our existing increment and decrement subroutines:
Using Our Subroutines

- We'll build our new machine using our existing increment and decrement subroutines:
Using Our Subroutines

- We'll build our new machine using our existing increment and decrement subroutines:

... 1 3 7 4 1 ...

...
Using Our Subroutines

- We'll build our new machine using our existing increment and decrement subroutines:
Using Our Subroutines

- We'll build our new machine using our existing increment and decrement subroutines:
Using Our Subroutines

- We'll build our new machine using our existing increment and decrement subroutines:
Using Our Subroutines

- We'll build our new machine using our existing increment and decrement subroutines:

... 1 3 7 4 1 ...
Using Our Subroutines

- We'll build our new machine using our existing increment and decrement subroutines:
Using Our Subroutines

- We'll build our new machine using our existing increment and decrement subroutines:

| ... | 1 | 3 | 7 | 4 | 1 | ... |

[Diagram showing a sequence of numbers with arrows indicating the flow or operation process]
Using Our Subroutines

- We'll build our new machine using our existing increment and decrement subroutines:
Using Our Subroutines

- We'll build our new machine using our existing increment and decrement subroutines:

```
... 1 3 8 4 1 ...
```
Using Our Subroutines

- We'll build our new machine using our existing increment and decrement subroutines:
Using Our Subroutines

- We'll build our new machine using our existing increment and decrement subroutines:
Using Our Subroutines

- We'll build our new machine using our existing increment and decrement subroutines:
0 → 0, R
1 → 1, R
...
9 → 9, R

start

To 2nd num

... 1 3 7 4 2 ...

...
| ... | 1 | 3 | 7 | 4 | 2 | ... |

The diagram shows a start state with transitions:
- $0 \rightarrow 0, R$
- $1 \rightarrow 1, R$
- $\ldots$
- $9 \rightarrow 9, R$
To 2\textsuperscript{nd} num

\begin{align*}
0 & \rightarrow 0, R \\
1 & \rightarrow 1, R \\
\ldots & \\
9 & \rightarrow 9, R
\end{align*}
0 → 0, R
1 → 1, R
...
9 → 9, R

start

To 2nd num

... 1 3 7 4 2 ...

...
To 2\textsuperscript{nd} num

\begin{itemize}
  \item $0 \rightarrow 0, R$
  \item $1 \rightarrow 1, R$
  \item $\ldots$
  \item $9 \rightarrow 9, R$
\end{itemize}

$\square \rightarrow \square, R$
0 → 0, R
1 → 1, R
...  
9 → 9, R

To 2nd num

□ → □, R
To 2\textsuperscript{nd} num

\begin{itemize}
\item 0 → 0, R
\item 1 → 1, R
\item \ldots
\item 9 → 9, R
\end{itemize}

\begin{itemize}
\item \(\square\rightarrow \square\), R
\end{itemize}

\textit{decr}
To 2\textsuperscript{nd} num

0 → 0, R
1 → 1, R
...
9 → 9, R

\textbf{start}

\[ \square \rightarrow \square, R \]

\textit{decr}

... 1 3 7 4 2 ...

...
0 → 0, R
1 → 1, R
... 
9 → 9, R

\[\text{start}\]

To 2\text{nd} num

\[\square \rightarrow \square, \ R\]

\text{decr}

... 1 3 7 4 2 ...
0 → 0, R
1 → 1, R
...
9 → 9, R

start

To 2\textsuperscript{nd} num

\[ \square \rightarrow \square, \ R \]

decr

... 1 3 7 4 2 ...

...
start

0 → 0, R
1 → 1, R
...
9 → 9, R

To 2\textsuperscript{nd} num

\[ \square \rightarrow \square, R \]

decr

\[ \cdots \]

\[ \begin{array}{c}
\cdots \\
1 \\
3 \\
7 \\
4 \\
2 \\
\end{array} \]
0 → 0, R
1 → 1, R
... 
9 → 9, R

start
To 2nd num

☐ → ☐, R
decr

... 1 3 7 4 1 ...
0 \rightarrow 0, R
1 \rightarrow 1, R
\ldots
9 \rightarrow 9, R

\text{start}

\text{To 2}\text{nd num}

\square \rightarrow \square, R

decr

\text{\ldots} 1 3 7 4 1 \text{\ldots}
0 → 0, R
1 → 1, R
... 
9 → 9, R

To 2\textsuperscript{nd} num

\[ \square \rightarrow \square, R \]

decr

... 1 3 7 \ 4 1 ...

...
start

0 → 0, R
1 → 1, R
...
9 → 9, R

to 2\textsuperscript{nd} num

□ → □, R

decr

done

... 1 3 7 4 1 ...

...
0 → 0, R
1 → 1, R
...
9 → 9, R

To 2\textsuperscript{nd} num

□ → □, R

decr

done

0 → 0, L
1 → 1, L
...
9 → 9, L

To 1\textsuperscript{st} num

... 1 3 7 4 1 ...

...
0 \rightarrow 0, R
1 \rightarrow 1, R
\ldots
9 \rightarrow 9, R

\rightarrow \square, R

start

To 2\textsuperscript{nd} num

\square \rightarrow \square, R

To 1\textsuperscript{st} num
decr

done

\square \rightarrow \square, L

0 \rightarrow 0, L
1 \rightarrow 1, L
\ldots
9 \rightarrow 9, L

... 1 3 7 4 1 ...
0 → 0, R
1 → 1, R

... 9 → 9, R

start

To 2nd num

□ → □, R

decr

done

To 1st num

0 → 0, L
1 → 1, L

... 9 → 9, L

□ → □, L

... 1 3 7 4 1 ...

...
To 2nd num

start

0 → 0, R
1 → 1, R
...
9 → 9, R

decr

To 1st num

done

Go home

0 → 0, L
1 → 1, L
...
9 → 9, L

... 1 3 7 4 1 ...
start

0 → 0, R
1 → 1, R
...
9 → 9, R

To 2nd num

□ → □, R

decr

done

0 → 0, L
1 → 1, L
...
9 → 9, L

To 1st num

□ → □, L

Go home

0 → 0, L
1 → 1, L
...
9 → 9, L

... 1 3 7 4 1 ...

...
0 → 0, R
1 → 1, R
...
9 → 9, R

To 2^{\text{nd}} \text{ num}

□ → □, R

\text{decr}

done

0 → 0, L
1 → 1, L
...
9 → 9, L

To 1^{\text{st}} \text{ num}

□ → □, L

0 → 0, L
1 → 1, L
...
9 → 9, L

\text{Go home}

□ → □, R

... 1 3 7 4 1 ...
start

0 → 0, R
1 → 1, R
...
9 → 9, R

To 2nd num

□ → □, R

decr

done

To 1st num

□ → □, L

0 → 0, L
1 → 1, L
...
9 → 9, L

incr

□ → □, R

Go home

□ → □, L

0 → 0, L
1 → 1, L
...
9 → 9, L

... 1 3 7 4 1 ...

...
0 → 0, R
1 → 1, R
...
9 → 9, R

start

To 2\textsuperscript{nd} num

\[
\begin{array}{c}
\square \rightarrow \square, R
\end{array}
\]

decr

done

0 → 0, L
1 → 1, L
...
9 → 9, L

To 1\textsuperscript{st} num

\[
\begin{array}{c}
\square \rightarrow \square, L
\end{array}
\]

incr

\[
\begin{array}{c}
\square \rightarrow \square, R
\end{array}
\]

Go home

\[
\begin{array}{c}
0 \rightarrow 0, L
1 \rightarrow 1, L
...
9 \rightarrow 9, L
\end{array}
\]

... 1 3 7 4 1 ...
To 2nd num
0 → 0, R
1 → 1, R
...
9 → 9, R

deincr

To 1st num
0 → 0, L
1 → 1, L
...
9 → 9, L

incr

Go home

... 1 3 7 4 1 ...
Go home

To 2nd num

0 → 0, R
1 → 1, R
...
9 → 9, R

To 1st num

0 → 0, L
1 → 1, L
...
9 → 9, L

Start

decr

To 1st num

incr

done

0 → 0, L
1 → 1, L
...
9 → 9, L

Go home

... 1 3 8 4 1 ...

...
Go home

To 2nd num

\[ \begin{align*}
0 & \rightarrow 0, R \\
1 & \rightarrow 1, R \\
... & \\
9 & \rightarrow 9, R
\end{align*} \]

\[ \quad \square \rightarrow \square, R \]

decr

done

To 1st num

\[ \begin{align*}
0 & \rightarrow 0, L \\
1 & \rightarrow 1, L \\
... & \\
9 & \rightarrow 9, L
\end{align*} \]

\[ \quad \square \rightarrow \square, L \]

incr

Go home

\[ \begin{align*}
0 & \rightarrow 0, L \\
1 & \rightarrow 1, L \\
... & \\
9 & \rightarrow 9, L
\end{align*} \]

\[ \quad \square \rightarrow \square, R \]

\[ \downarrow \]

... 1 3 8 4 1 ...
To 2nd num

0 → 0, R
1 → 1, R
...
9 → 9, R

To 1st num

0 → 0, L
1 → 1, L
...
9 → 9, L

Start

 incr

 done

 Go home

 decr

 done

... 1 3 8 4 1 ...

...
To 2nd num

0 → 0, R
1 → 1, R
...
9 → 9, R

To 1st num

0 → 0, L
1 → 1, L
...
9 → 9, L

incr

0 → 0, L
1 → 1, L
...
9 → 9, L

decr

done

0 → 0, R
1 → 1, R
...
9 → 9, R

Go home

done

... 1 3 8 4 1 ...

...
To 2\textsuperscript{nd} num

\begin{align*}
0 &\rightarrow 0, R \\
1 &\rightarrow 1, R \\
\ldots \\
9 &\rightarrow 9, R
\end{align*}

decr

\begin{align*}
\underline{\square} &\rightarrow \underline{\square}, R
\end{align*}

done

\begin{align*}
0 &\rightarrow 0, L \\
1 &\rightarrow 1, L \\
\ldots \\
9 &\rightarrow 9, L
\end{align*}

To 1\textsuperscript{st} num

\begin{align*}
\underline{\square} &\rightarrow \underline{\square}, L
\end{align*}

incr

\begin{align*}
\underline{\square} &\rightarrow \underline{\square}, R
\end{align*}

Go home

\begin{align*}
0 &\rightarrow 0, L \\
1 &\rightarrow 1, L \\
\ldots \\
9 &\rightarrow 9, L
\end{align*}

Done

\begin{align*}
\rightarrow
\end{align*}

... 1 3 8 4 1...

...
... 1 3 8 4 0 ...

0 → 0, R
1 → 1, R
... 9 → 9, R

0 → 0, L
1 → 1, L
... 9 → 9, L

To 1st num

To 2nd num

Home

Go

Start

done

done

incr

decr
start

0 → 0, R
1 → 1, R
...
9 → 9, R

To 2ⁿᵈ num

□ → □, R

decr

□ → □, L

To 1ˢᵗ num

0 → 0, L
1 → 1, L
...
9 → 9, L

done

□ → □, L

居家

□ → □, R

Go home

done

0 → 0, L
1 → 1, L
...
9 → 9, L

0 → 0, R
1 → 1, R
...
9 → 9, R

... 1 3 8 4 0 ...

...
To 1
Go home
decr
To 2
To 1st
num
num

\[
\begin{array}{c}
0 \rightarrow 0, R \\
1 \rightarrow 1, R \\
\vdots \\
9 \rightarrow 9, R \\
\end{array}
\]

\[
\begin{array}{c}
\square \rightarrow \square, R \\
\square \rightarrow \square, L \\
\end{array}
\]

\[
\begin{array}{c}
0 \rightarrow 0, L \\
1 \rightarrow 1, L \\
\vdots \\
9 \rightarrow 9, L \\
\end{array}
\]

\[
\begin{array}{c}
0 \rightarrow 0, L \\
1 \rightarrow 1, L \\
\vdots \\
9 \rightarrow 9, L \\
\end{array}
\]
To 1st num

To 2nd num

start

0 → 0, R
1 → 1, R
...
9 → 9, R

While done

To 2nd num

To 1st num

0 → 0, R
1 → 1, R
...
9 → 9, R

While incr

0 → 0, L
1 → 1, L
...
9 → 9, L

While decr

0 → 0, R
1 → 1, R
...
9 → 9, R

Done

... 1 3 8 4 0 ...

...
start

0 → 0, R
1 → 1, R
...
9 → 9, R

To 2nd num

□ → □, R
decr

done

To 1st num

□ → □, L

incr

□ → □, R

Go home

0 → 0, L
1 → 1, L
...
9 → 9, L

done

0 → 0, L
1 → 1, L
...
9 → 9, L
To 2\textsuperscript{nd} num

0 → 0, R
1 → 1, R
...
9 → 9, R

start

To 1\textsuperscript{st} num
doing

0 → 0, L
1 → 1, L
...
9 → 9, L

done

To 2\textsuperscript{nd} num

0 → 0, R
1 → 1, R
...
9 → 9, R

To 1\textsuperscript{st} num

0 → 0, L
1 → 1, L
...
9 → 9, L

done

... 1 3 9 4 0 ...
...
To 1st num

0 → 0, R
1 → 1, R
...
9 → 9, R

To 2nd num

□ → □, R

 incr
done

0 → 0, L
1 → 1, L
...
9 → 9, L
done

Go home

□ → □, L

decr

□ → □, R

... 1 3 9 3 9 ...

...
To 1st num

0 → 0, R
1 → 1, R
...
9 → 9, R

To 2nd num

0 → 0, R
1 → 1, R
...
9 → 9, R

incr

To 1st num

0 → 0, L
1 → 1, L
...
9 → 9, L

done

home

0 → 0, L
1 → 1, L
...
9 → 9, L

done

start

... 1 3 9 3 9 ...

...
To 1st num

To 2nd num

Go home

decr

done

incr

0 → 0, R
1 → 1, R
...
9 → 9, R

0 → 0, L
1 → 1, L
...
9 → 9, L

... 1 3 9 3 9 ...

...
0 → 0, R
1 → 1, R
...
9 → 9, R

To 2\textsuperscript{nd} num

\[ \square \rightarrow \square, \text{R} \]

decr

To 1\textsuperscript{st} num

\[ \square \rightarrow \square, \text{R} \]

\[ \square \rightarrow \square, \text{L} \]

done

0 → 0, L
1 → 1, L
...
9 → 9, L

Go home

0 → 0, L
1 → 1, L
...
9 → 9, L

home

\[ \square \rightarrow \square, \text{R} \]

To 2\textsuperscript{nd} num

\[ \square \rightarrow \square, \text{R} \]

\[ \square \rightarrow \square, \text{L} \]

done

\[ \square \rightarrow \square, \text{L} \]

0 → 0, L
1 → 1, L
...
9 → 9, L

... 1 3 9 3 9 ...

...
Go home

To 1st num

Go home

done

To 2nd num

1 3 9 3 9

...
Many transitions later...
To 1st num

To 2nd num

deincr
done
done

0 → 0, R
1 → 1, R
...
9 → 9, R

0 → 0, L
1 → 1, L
...
9 → 9, L

... 1 7 8 0 0 ...
To 2nd num

0 → 0, R
1 → 1, R
...
9 → 9, R

To 1st num

0 → 0, L
1 → 1, L
...
9 → 9, L

Go home

0 → 0, L
1 → 1, L
...
9 → 9, L

 incr
done

 decr
done

start

... 1 7 9 0 0 ...

...
To 2\textsuperscript{nd} num \quad \text{done!} \quad \text{To 1\textsuperscript{st} num} 

\begin{align*}
0 &\rightarrow 0, R \\
1 &\rightarrow 1, R \\
\vdots & \\
9 &\rightarrow 9, R \\
\underline{17900} &
\end{align*}
To 2\textsuperscript{nd} num

\begin{itemize}
\item 0 → 0, R
\item 1 → 1, R
\item \ldots
\item 9 → 9, R
\end{itemize}

\textit{start}

done!

\texttt{decr}

\begin{itemize}
\item n = 0
\end{itemize}

done

\texttt{To 1\textsuperscript{st}
num}

\begin{itemize}
\item 0 → 0, L
\item 1 → 1, L
\item \ldots
\item 9 → 9, L
\end{itemize}

\texttt{Go home}

\begin{itemize}
\item 0 → 0, L
\item 1 → 1, L
\item \ldots
\item 9 → 9, L
\end{itemize}

\texttt{incr}

\begin{itemize}
\item \textit{done}
\item \textit{done}
\end{itemize}

\texttt{home}

\begin{itemize}
\item \textit{done}
\item \textit{done}
\end{itemize}
Using Subroutines

- Once you've built a subroutine, you can wire it into another TM with something that, schematically, looks like this:

  ![Diagram](image)

- Intuitively, this corresponds to transitioning to the start state of the subroutine, then replacing the “done” state of the subroutine with the state at the end of the transition.
Bonus Fun Read

- Check out the short story "The Feeling of Power" by Isaac Asimov, set in a world where people have forgotten how to add and subtract.