Verifiers and RE
Get ready to answer some questions in rapid-fire style!
(about 10 seconds per question, but I won’t close the recorded poll for a while so don’t stress about that)
Definition:

A **k-Clique** is a set of $k$ vertices of a graph that are all adjacent to each other (all possible edges between those $k$ vertices are present in the graph).

*has a 4-Clique:*

*does not have a 4-Clique (has 3-Clique though):*
QUICK REACTION: Does this graph contain a 4-clique?

Answer at PollEv.com/cs103 or text CS103 to 22333 once to join, then Y, N, or ? (for “I don’t know”).
WITH A HINT: Does this graph contain a 4-clique?

Answer at PollEv.com/cs103 or text CS103 to 22333 once to join, then Y, N, or ? (for “I don’t know”).
WITH A NEW HINT: Does this graph contain a 4-clique?

Answer at PollEv.com/cs103 or text CS103 to 22333 once to join, then Y, N, or ? (for “I don’t know”).
Key Intuition:

A language $L$ is in RE if, for any string $w$, if you are convinced that $w \in L$, there is some piece of evidence you could provide to convince someone else.
Discussion Question:

A language $L$ is in $\text{RE}$ if, for any string $w$, if you are convinced that $w \in L$, there is some piece of evidence you could provide to convince someone else.

What about for a $w \notin L$? What would a piece of evidence for that look like?
More examples of helpful hints vs unhelpful hints
Does this Sudoku puzzle have a solution?
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Does this Sudoku puzzle have a solution?
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Verification

Does this graph have a Hamiltonian path (a simple path that passes through every node exactly once?)
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Verification

Does this graph have a *Hamiltonian path* (a simple path that passes through every node exactly once?)
Does this graph have a Hamiltonian path (a simple path that passes through every node exactly once?)
Verification

• In each of the preceding cases, we were given some problem and some evidence supporting the claim that the answer is “yes.”

• Given the correct evidence, we can be certain that the answer is indeed “yes.”

• Given incorrect evidence, we aren't sure whether the answer is “yes.”
  • Maybe there's no evidence saying that the answer is “yes,” because the answer is no!
  • Or maybe there is some evidence, but just not the evidence we were given.

• Let's formalize this idea.
Verifiers

- A **verifier** for a language $L$ is a TM $V$ with the following properties:
  - $V$ halts on all inputs.
  - For any string $w \in \Sigma^*$, the following is true:
    \[
    w \in L \iff \exists c \in \Sigma^*. \ V \text{ accepts } \langle w, c \rangle
    \]
- A string $c$ where $V$ accepts $\langle w, c \rangle$ is called a **certificate** for $w$.
- Intuitively, what does this mean?
Deciders and Verifiers

Decider $M$ for $L$

M halts on all inputs.
$w \in L \iff M$ accepts $w$

Verifier $V$ for $L$

V halts on all inputs.
$w \in L \iff \exists c \in \Sigma^* . V$ accepts $(w, c)$

“Solve the problem”

If $M$ accepts, then $w \in L$.

If $M$ rejects, then $w \notin L$.

“Check the answer”

If $V$ accepts $(w, c)$, then $w \in L$.

If $V$ rejects $(w, c)$, we don't know whether $w \in L$. 
Deciders and Verifiers

Decider $M$ for $L$

$M$ halts on all inputs. $w \in L \leftrightarrow M$ accepts $w$

Verifier $V$ for $L$

$V$ halts on all inputs. $w \in L \leftrightarrow \exists c \in \Sigma^*. V$ accepts $(w, c)$

If $M$ accepts, then $w \in L$.
If $M$ rejects, then $w \notin L$.

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- $V$ halts on all inputs.
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If $M$ accepts, then $w \in L$.

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Deciders and Verifiers

Decider $M$ for $L$

- $M$ halts on all inputs. $w \in L \iff M$ accepts $w$
- "Solve the problem"

Verifier $V$ for $L$

- $V$ halts on all inputs. $w \in L \iff \exists c \in \Sigma^*. V$ accepts $(w, c)$
- "Check the answer"

If $M$ accepts, then $w \in L$. 
If $M$ rejects, then $w \notin L$. 
If $V$ accepts $(w, c)$, then $w \in L$. 
If $V$ rejects $(w, c)$, we don't know whether $w \in L$. 
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Input string ($w$)
Certificate ($c$)

Deciders and Verifiers

**Decider** $M$ for $L$

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"Solve the problem"

"Check the answer"
Verifiers

• A **verifier** for a language $L$ is a TM $V$ with the following properties:
  
  • $V$ halts on all inputs.
  
  • For any string $w \in \Sigma^*$, the following is true:
    
    $w \in L \iff \exists c \in \Sigma^*. V$ accepts $\langle w, c \rangle$

• Some notes about $V$:
  
  • If $V$ accepts $\langle w, c \rangle$, then we're guaranteed $w \in L$.
  
  • If $V$ does not accept $\langle w, c \rangle$, then either
    
    – $w \in L$, but you gave the wrong $c$, or
    
    – $w \not\in L$, so no possible $c$ will work.
Verifiers

- A **verifier** for a language $L$ is a TM $V$ with the following properties:
  - $V$ halts on all inputs.
  - For any string $w \in \Sigma^*$, the following is true:
    \[ w \in L \iff \exists c \in \Sigma^*. V \text{ accepts } \langle w, c \rangle \]

- More notes about $V$:
  - Notice that $c$ is existentially quantified.
  - Notice $V$ is required to halt *always* (like a decider).
Verifiers

A **verifier** for a language $L$ is a TM $V$ with the following properties:

- $V$ halts on all inputs.
- For any string $w \in \Sigma^*$, the following is true:
  \[ w \in L \iff \exists c \in \Sigma^*. V \text{ accepts } \langle w, c \rangle \]

More notes about $V$:

- Notice that $\mathcal{L}(V) \neq L$. *(Good question to hold on to for a second: what is $\mathcal{L}(V)$?)*
- The job of $V$ is just to check certificates, not to decide membership in $L$. 
Verifiers

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  - $V$ halts on all inputs.
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    $$w \in L \iff \exists c \in \Sigma^*. \; V \text{ accepts } \langle w, c \rangle$$

- A note about $c$:
  - Figuring out what would make a good certificate (should it be a number of steps to take, an equation-solving variable assignment, a set of graph nodes, an array of numbers to fill in a whole Sudoku board?) is custom work to do for each different language $L$. 
Some Verifiers

• Let $L$ be the following language:

$$
L = \{ \langle n \rangle \mid n \in \mathbb{N} \text{ and the hailstone sequence terminates for } n \}
$$

```cpp
bool checkHailstone(int n, int c) {
    for (int i = 0; i < c; i++) {
        if (n % 2 == 0) n /= 2;
        else n = 3*n + 1;
        if (n == 1) return true;
    }
    return n == 1;
}
```
Some Verifiers

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```

Does this always halt?

Answer at PollEv.com/cs103 or text CS103 to 22333 once to join, then Y or N.
Some Verifiers

For one given \( \langle n \rangle \in L \) (say 11), how many different values of \( c \) will work to cause the verifier to accept?

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```

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Some Verifiers

- \( L(V) = L \)
- \( L(V) \subseteq L \)
- \( L \subseteq L(V) \)

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    }
    return n == 1;
}
```

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Some Verifiers

• Let $L$ be the following language:

$$L = \{ \langle G \rangle \mid G \text{ is a graph and } G \text{ has a Hamiltonian path } \}$$

• (A Hamiltonian path is a simple path that visits every node in the graph.)

• Let's see how to build a verifier for $L$. 
Verification

Is there a simple path that goes through every node exactly once?
Verifier Example: Hamiltonian Path

- Let $L$ be the following language:

$$L = \{ \langle G \rangle \mid G \text{ is a graph with a Hamiltonian path} \}$$

```cpp
bool checkHamiltonian(Graph G, vector<Node> c) {
    if (c.size() != G.numNodes()) return false;
    if (containsDuplicate(c)) return false;
    for (size_t i = 0; i < c.size() - 1; i++) {
        if (!G.hasEdge(c[i], c[i+1])) return false;
    }
    return true;
}
```

- Do you see why $\langle G \rangle \in L$ iff there is a $c$ where $\text{checkHamiltonian}(G, c)$ returns true?

- Do you see why $\text{checkHamiltonian}$ always halts?
Where We’ve Been

State Elimination

NFA  -->  Regex

Thompson’s Algorithm
Where We’re Going Today

Verifier

Recognizer

Somehow build this

Somehow build this
Verifier for $A_{TM}$?

- Consider $A_{TM}$:
  
  $$A_{TM} = \{ \langle M, w \rangle \mid M \text{ is a TM and } M \text{ accepts } w \}.$$  

- This is a **canonical** example of an undecidable language. There’s no way, in general, to tell whether a TM $M$ will accept a string $w$.

- Although this language is undecidable, it’s an **RE** language, and it’s possible to build a verifier for it!
What would make a good certificate for a verifier for $A_{TM}$?

- Consider $A_{TM}$:

  \[ A_{TM} = \{ \langle M, w \rangle \mid M \text{ is a TM and } M \text{ accepts } w \} \].

- This is a *canonical* example of an undecidable language. There’s no way, in general, to tell whether a TM $M$ will accept a string $w$.

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Answer at PollEv.com/cs103 or text CS103 to 22333 once to join, then an idea
Run this TM for fifteen steps.
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Run this TM for fifteen steps.
Run this TM for fifteen steps.
0 → 0, R
☐ → ☐, R

0 → 0, L
1 → 1, L

1 → ☐, L

☐ → ☐, L

Run this TM for fifteen steps.

... 0 1 ...
Run this TM for fifteen steps.
Run this TM for fifteen steps.
Run this TM for fifteen steps.
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Run this TM for fifteen steps.
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Run this TM for fifteen steps.
A Verifier for $A_{TM}$

- Recall $A_{TM} = \{ (M, w) \mid M$ is a TM and $M$ accepts $w \}$

```cpp
def checkWillAccept(TM M, string w, int c):
    set up a simulation of M running on w;
    for (int i = 0; i < c; i++) {
        simulate the next step of M running on w;
    }
    return whether M is in an accepting state;
```

- Do you see why $M$ accepts $w$ iff there is some $c$ such that checkWillAccept($M$, $w$, $c$) returns true?
- Do you see why checkWillAccept always halts?
Equivalence of Verifiers and Recognizers

Verifier

Recognizer

Enforce a step count
What languages are verifiable?
Let $V$ be a verifier for a language $L$. Consider the following function given in pseudocode:

```c
bool mysteryFunction(string w) {
    int i = 0;
    while (true) {
        for (each string c of length i) {
            if (V accepts $\langle w, c \rangle$) return true;
        }
        i++;
    }
}
```

What set of strings does `mysteryFunction` return `true` on?
Equivalence of Verifiers and Recognizers

Verifier

Try all certificates

Recognizer

Enforce a step count
**Theorem:** If $L$ is a language, then there is a verifier for $L$ if and only if $L \in \text{RE.}$
Verifiers and \textbf{RE}

- **Theorem:** If there is a verifier $V$ for a language $L$, then $L \in \textbf{RE}$.

- **Proof goal:** Given a verifier $V$ for a language $L$, find a way to construct a recognizer $M$ for $L$.  

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![Diagram](image-url)
Theorem: If there is a verifier $V$ for a language $L$, then $L \in \text{RE}$.

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We will try all possible certificates (values of $c$)
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$\varepsilon$ $a$ $b$ $aa$ $ab$ $ba$ $bb$ $aaa$ $aab$ $aba$ $abb$ $baa$ ...

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---

**Diagram:**
- **Verifier $V$ for $L$**
  - **input string** $(w)$
  - **certificate** $(c)$
- **“Check the answer”**
  - **yes!**
  - **not sure**

*We will try all possible certificates (values of $c$)*

- $\varepsilon$, a, b, aa, ab, ba, bb, aaa, aab, aba, abb, baa, ...

---

**Verifiers and RE**
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```
\epsilon \ a \ b \ aa \ ab \ ba \ bb \ aaa \ aab \ aba \ abb \ baa \ ...
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\begin{itemize}
  \item \textit{Verifier} $V$ for $L$
  \item \textit{Check the answer}:
  \begin{itemize}
    \item \text{input string} (w)
    \item \text{certificate} (c)
  \end{itemize}
  \end{itemize}

\textit{We will try all possible certificates (values of c)}

\[
\begin{array}{cccccccccccc}
\varepsilon & a & b & aa & ab & ba & bb & aaa & aab & aba & abb & baa & \ldots \\
\end{array}
\]
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$\epsilon$  a  b  aa  ab  ba  bb  aaa  aab  aba  abb  baa  ...
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Verifiers and RE
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*We will try all possible certificates (values of $c$)*
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We will try all possible certificates (values of $c$)

\begin{itemize}
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  \item a
  \item b
  \item aa
  \item ab
  \item ba
  \item bb
  \item aaa
  \item aab
  \item aba
  \item abb
  \item baa
  \item \ldots
\end{itemize}
Verifiers and \textbf{RE}

\begin{itemize}
  \item \textbf{Theorem:} If there is a verifier \( V \) for a language \( L \), then \( L \in \text{RE} \).
  \item \textbf{Proof goal:} Given a verifier \( V \) for a language \( L \), find a way to construct a recognizer \( M \) for \( L \).
\end{itemize}

We will try all possible certificates (values of \( c \))

\[ \varepsilon \ a \ b \ aa \ ab \ ba \ bb \ aaa \ aab \ aba \ abb \ baa \ ... \]
Verifiers and RE

- **Theorem:** If there is a verifier $V$ for a language $L$, then $L \in \text{RE}$.

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We will try all possible certificates (values of $c$):

- $\varepsilon$
- $a$
- $b$
- $aa$
- $ab$
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- $aaa$
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- $abb$
- $baa$
- ...
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\begin{align*}
\varepsilon & \quad a & \quad b & \quad aa & \quad ab & \quad ba & \quad bb & \quad aaa & \quad aab & \quad aba & \quad abb & \quad baa & \quad ... 
\end{align*}
Verifiers and \textbf{RE}

- **Theorem:** If there is a verifier \( V \) for a language \( L \), then \( L \in \text{RE} \).

- **Proof goal:** Given a verifier \( V \) for a language \( L \), find a way to construct a recognizer \( M \) for \( L \).

We will try all possible certificates (values of \( c \)):
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| $\varepsilon$ | a | b | aa | ab | ba | bb | aaa | aab | aba | abb | baa | ...
|----------------|---|---|----|----|----|----|-----|-----|-----|-----|-----|-----|

"Check the answer"
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\[\varepsilon \quad a \quad b \quad aa \quad ab \quad ba \quad bb \quad aaa \quad aab \quad aba \quad abb \quad baa \quad \ldots\]
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**Diagram:**

- **Verifier $V$ for $L$:**
  - **Input string ($w$):**
  - **Certificate ($c$):**
  - "Check the answer"
  - If the answer is "yes!" then $w \in L$.
  - If the answer is "not sure" then the certificate is invalid.

**We will try all possible certificates ($\text{values of } c$):**

- $\varepsilon$
- $a$
- $b$
- $aa$
- $ab$
- $ba$
- $bb$
- $aaa$
- $aab$
- $aba$
- $abb$
- $baa$
- $...$
Verifiers and RE

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$\varepsilon$  a  b  aa  ab  ba  bb  aaa  aab  aba  abb  baa  ...
Verifiers and RE

• **Theorem:** If $V$ is a verifier for $L$, then $L \in \text{RE}$.

• **Proof sketch:** Consider the following program:

```cpp
bool isInL(string w) {
  int i = 0;
  while (true) {
    for (each string $c$ of length $i$) {
      if ($V$ accepts $\langle w, c \rangle$) return true;
    }
    i++;
  }
}
```

If $w \in L$, there is some $c \in \Sigma^*$ where $V$ accepts $\langle w, c \rangle$. The function `isInL` tries all possible strings as certificate, so it will eventually find $c$ (or some other certificate), see $V$ accept $\langle w, c \rangle$, then return true. Conversely, if `isInL(w)` returns true, then there was some string $c$ such that $V$ accepted $\langle w, c \rangle$, so $w \in L$. ■
Verifiers and $\text{RE}$

- **Theorem:** If $L \in \text{RE}$, then there is a verifier for $L$.
- **Proof goal:** Beginning with a recognizer $M$ for the language $L$, show how to construct a verifier $V$ for $L$.

- The challenges:
  - A recognizer $M$ is not required to halt on all inputs. A verifier $V$ must always halt.
  - A recognizer $M$ takes in one single input. A verifier $V$ takes in two inputs.
- We’ll need to find a way of reconciling these requirements.
Recall: If $M$ is a recognizer for a language $L$, then $M$ accepts $w$ iff $w \in L$.

Key insight: If $M$ accepts a string $w$, it always does so in a finite number of steps.

Idea: Adapt the verifier for $A_{TM}$ into a more general construction that turns any recognizer into a verifier by running it for a fixed number of steps.
Verifiers and $\text{RE}$

• **Theorem:** If $L \in \text{RE}$, then there is a verifier for $L$.

• **Proof sketch:** Consider the following program:

```cpp
bool checkIsInL(string w, int c) {
    set up a simulation of M running on w;
    for (int i = 0; i < c; i++) {
        simulate the next step of M running on W;
    }
    return whether M is in an accepting state;
}
```

Notice that `checkIsInL` always halts, since each step takes only finite time to complete. Next, notice that if there is a $c$ where `checkIsInL(w, c)` returns true, then $M$ accepted $w$ after running for $c$ steps, so $w \in L$. Conversely, if $w \in L$, then $M$ accepts $w$ after some number of steps (call that number $c$). Then `checkIsInL(w, c)` will run $M$ on $w$ for $c$ steps, watch $M$ accept $w$, then return true. ■
RE and Proofs

- Verifiers and recognizers give two different perspectives on the “proof” intuition for RE.
- Verifiers are explicitly built to check proofs that strings are in the language.
  - If you know that some string $w$ belongs to the language and you have the proof of it, you can convince someone else that $w \in L$.
- You can think of a recognizer as a device that “searches” for a proof that $w \in L$.
  - If it finds it, great!
  - If not, it might loop forever.
RE and Proofs

• If the RE languages represent languages where membership can be proven, what does a non-RE language look like?

• Intuitively, a language is *not* in RE if there is no general way to prove that a given string $w \in L$ actually belongs to $L$.

• In other words, even if you knew that a string was in the language, you may never be able to convince anyone of it!