The Guide to Self-Reference
Hi everybody!
Self-reference proofs can be pretty hard to understand the first time you see them.
If you're confused - that's okay! It's totally normal. This stuff is tricky.
Once you get a better sense for how to structure these proofs, I think you’ll find that they’re not as bad as they initially seem.
What does this program do?

```cpp
bool willAccept(string program, string input) {
    /* ... some implementation ... */
}

int main() {
    string me = mySource();
    string input = getInput();
    if (willAccept(me, input)) {
        reject();
    } else {
        accept();
    }
}
```

This lecture slide was the first time that we really saw self-reference, and a lot of you got pretty tripped up by what was going on.

Try running this program on any input.

What happens if

... this program accepts its input?
   It rejects the input!

... this program doesn't accept its input?
   It accepts the input!
What does this program do?

```c++
bool willAccept(string program, string input) {
    /* ... some implementation ... */
}

int main() {
    string me = mySource();
    string input = getInput();
    if (willAccept(me, input)) {
        reject();
    } else {
        accept();
    }
}
```

Part of the reason why this can be tricky is that what you're looking at is a finished product. If you don't have a sense of where it comes from, it's really hard to understand!

Try running this program on any input. What happens if...

... this program accepts its input?
It rejects the input!

... this program doesn't accept its input?
It accepts the input!
What does this program do?

```cpp
bool willAccept(string program, string input) {
    /* … some implementation … */
}

int main() {
    string me = mySource();
    string input = getInput();

    if (willAccept(me, input)) {
        reject();
    } else {
        accept();
    }
}
```

Try running this program on any input. What happens if

... this program accepts its input?
It rejects the input!

... this program doesn't accept its input?
It accepts the input!

Let's see where it comes from!

We'll take it from the top.
Let's try to use self-reference to prove that $A_{TM}$ is undecidable.
At a high level, we're going to do a proof by contradiction.
We’re going to start off by assuming that $A_{TM}$ is decidable.
Somehow, we’re going to try to use this to get to a contradiction.
If we can get a contradiction — any contradiction — we’ll see that our assumption was wrong.
The challenge is figuring out exactly how to go and do this.
Rather than just jumping all the way to the end, let's see what our initial assumption tells us.
We’re assuming that $A_{TM}$ is decidable. What does that mean?
Well, a language is decidable if there's a decider for it, so that means there's some decider for $A_{TM}$. Let's call that decider $D$. Contradiction!
There is a decider $D$ for $A_{TM}$

What might this decider look like?

Contradiction!
A decider for a language is a Turing machine with a few key properties.

Contradiction!
There is a decider $D$ for $A_{TM}$

First, it has to always halt.

Contradiction!
There is a decider $D$ for $A_{TM}$.

That means that if you give it any input, it has to either accept or reject it. We'll visualize this with these two possible outputs.

Contradiction!
There is a decider $D$ for $A_{TM}$.

Next, the decider has to tell us something about $A_{TM}$.

Contradiction!
$A_{TM} \in \mathbb{R}$

There is a decider $D$ for $A_{TM}$

Next, the decider has to tell us something about $A_{TM}$.

As a reminder, $A_{TM}$ is the language $\{ \langle M, w \rangle \mid M$ is a TM and $M$ accepts $w \}$.
Specifically, the decider $D$ needs to take in an input and tell us whether that input is in $A_{TM}$.

As a reminder, $A_{TM}$ is the language

$$\{ \langle M, w \rangle \mid M \text{ is a TM and } M \text{ accepts } w \}$$
There is a decider $D$ for $A_{TM}$.

$A_{TM} \in \mathbb{R}$

$A_{TM}$ is a language of pairs of TMs and strings, so $D$ will take in two inputs, a machine $M$ and a string $w$.

As a reminder, $A_{TM}$ is the language

$\{ \langle M, w \rangle | M \text{ is a TM and } M \text{ accepts } w \}$

Contradiction!
$A_{TM} \in \mathbb{R}$

There is a decider $D$ for $A_{TM}$

Decider $D$ for $A_{TM}$

If $D$ accepts its input, it means that $\langle M, w \rangle$ is in $A_{TM}$.

Yes, $M$ accepts $w$.

As a reminder, $A_{TM}$ is the language

\{ $\langle M, w \rangle$ | $M$ is a TM and $M$ accepts $w$ \}
There is a decider \( D \) for \( A_{\text{TM}} \).

As a reminder, \( A_{\text{TM}} \) is the language:

\[
\{ \langle M, w \rangle \mid M \text{ is a TM and } M \text{ accepts } w \}.
\]

Contradiction!

Otherwise, if \( D \) rejects its input, it means that \( M \) doesn't accept \( w \).

Yes, \( M \) accepts \( w \).

No, \( M \) does not accept \( w \).
So now we’ve got this TM $D$ lying around. What can we do with it?

Contradiction!
We've seen the idea that TMs can run other TMs as subroutines. This means we can write programs that use $D$ as a subroutine.

Contradiction!

There is a decider $D$ for $A_{TM}$

We can write programs that use $D$ as a helper method.
Since TMs are kinda like programs, we can imagine that $D$ is a helper method that looks like this.

There is a decider $D$ for $A_{TM}$.

We can write programs that use $D$ as a helper method.

Contradiction!
In mathematics, the convention is to use single-letter variable names for everything, which isn’t good programming style.

Contradiction!
Here, the method name (willAccept) is just a fancier and more descriptive name for $D$.
$A_{TM} \in \mathbb{R}$

There is a decider $D$ for $A_{TM}$

We can write programs that use $D$ as a helper method

Contradiction!

The two arguments to $\text{willAccept}$ then correspond to the inputs to the decider $D$.
When thinking of $D$ as a decider, we think of it accepting or rejecting. In programming-speak, it's like returning a boolean.

Contradiction!
So at this point we've just set up the fact that this subroutine exists. What exactly are we going to do with it?

Contradiction!
Ultimately, we're trying to get a contradiction.

There is a decider $D$ for $A_{TM}$

We can write programs that use $D$ as a helper method

Contradiction!
Specifically, we’re going to build a program – which we’ll call $P$ – that has some really broken behavior... it will accept its input if and only if it doesn’t accept its input!

Contradiction!
If you're wondering how on earth you were supposed to figure out that that's the next step, don't panic. The first time you see it, it looks totally crazy. Once you've done this a few times, you'll get a lot more comfortable with it.

Contradiction!
Now, we haven't actually written this program $P$ yet. That's the next step.

Contradiction!
If you look at what we've said, right now we have a goal of what \( P \) should do, not how \( P \) actually does that.

Contradiction!

There is a decider \( D \) for \( A_{TM} \).

We can write programs that use \( D \) as a helper method.

Program \( P \) accepts its input if and only if \( P \) does not accept its input.

If \( M \) accepts \( w \), then \( M \) also accepts \( w \).

If \( M \) does not accept \( w \), then \( M \) does not accept \( w \).

We can write a program `willAccept` that takes a program and an input as arguments:

```cpp
bool willAccept(string program, string input) {
    // Use the decider D to accept or reject the input.
    // The program P accepts input if and only if willAccept returns true.
    // The program P rejects input if and only if willAccept returns false.
}
```
A_{TM} \in \mathbb{R}

There is a decider \( D \) for \( A_{TM} \)

We can write programs that use \( D \) as a helper method

\textbf{Program } \( P \) accepts its input if and only if program \( P \) does not accept its input

Contradiction!

\textbf{bool willAccept(string program, string input)}

You can think of this requirement as a sort of "design specification."
Let's actually go write out a spec for what $P$ needs to do!

There is a decider $D$ for $A_{TM}$

We can write programs that use $D$ as a helper method

Program $P$ design specification:

```cpp
bool willAccept(string program, string input)
```

Let's actually go write out a spec for what $P$ needs to do!
Since this requirement is an "if and only if," we can break it down into two cases.

Program $P$ design specification:

```cpp
bool willAccept(string program, string input)
```

We can write programs that use $D$ as a helper method.

Contradiction!
First, if this program $P$ is supposed to accept its input, then it needs to not accept its input. Contradiction!

There is a decider $D$ for $A_{TM}$.

We can write programs that use $D$ as a helper method.

Program $P$ design specification:

If $P$ accepts its input, then $P$ does not accept its input.

```python
bool willAccept(string program, string input)
```

Yes, $M$ accepts $w$.

No, $M$ does not accept $w$. 
Next, if this program $P$ is supposed to not accept its input, then it needs to accept its input.

Contradiction!

Program $P$ design specification:
- If $P$ accepts its input, then $P$ does not accept its input.
- If $P$ does not accept its input, then $P$ accepts its input.
There is a decider $D$ for $A_{TM}$.

We can write programs that use $D$ as a helper method.

Program $P$ design specification:
- If $P$ accepts its input, then $P$ does not accept its input.
- If $P$ does not accept its input, then $P$ accepts its input.

We now have a specification for what program $P$ is supposed to do. Let’s see how to write it!
There is a decider $D$ for $A_{TM}$

We can write programs that use $D$ as a helper method

Program $P$ accepts its input if and only if program $P$ does not accept its input

Contradiction!
There is a decider $D$ for $A_{TM}$

We can write programs that use $D$ as a helper method

Program $P$ accepts its input if and only if program $P$ does not accept its input

Contradiction!

$A_{TM} \in R$

Like most programs, our program begins execution in main().

```
// Program P

int main() {

    // ... (program code)

    // ... (program code)

    if (willAccept(me, input)) {
        reject();
    } else {
        accept();
    }

}
```
There is a decider $D$ for $A_{TM}$

We can write programs that use $D$ as a helper method

Program $P$ accepts its input if and only if program $P$ does not accept its input

Contradiction!

$A_{TM} \in \mathbb{R}$

Our program needs to get some input, so let's do that here.
There is a decider $D$ for $A_{TM}$.

We can write programs that use $D$ as a helper method.

Program $P$ accepts its input if and only if program $P$ does not accept its input.

Contradiction!
$A_{TM} \in R$

There is a decider $D$ for $A_{TM}$

We can write programs that use $D$ as a helper method

Program $P$ accepts its input if and only if program $P$ does not accept its input

Contradiction!

$\text{bool willAccept(string program, string input)}$

Program $P$ design specification:

If $P$ accepts its input, then

$P$ does not accept its input.

If $P$ does not accept its input, then

$P$ accepts its input.

// Program P
int main() {
    string input = getInput();
    string me = mySource();
    if (willAccept(me, input)) {
        reject();
    } else {
        accept();
    }
}

That means we need to be able to figure out whether we're going to accept.
There is a decider \( D \) for \( A_{TM} \)

We can write programs that use \( D \) as a helper method

Program \( P \) accepts its input if and only if program \( P \) does not accept its input

Contradiction!

\[ A_{TM} \in R \]

We've got this handy method lying around that will let us know whether any program will accept any input.
There is a decider $D$ for $A_{TM}$

We can write programs that use $D$ as a helper method

Program $P$ accepts its input if and only if program $P$ does not accept its input.

Contradiction!
A_{TM} \in \mathbb{R}

There is a decider $D$ for $A_{TM}$

We can write programs that use $D$ as a helper method

Program $P$ accepts its input if and only if program $P$ does not accept its input

Contradiction!

$\forall M \in \mathbb{R}$

Decider $D$ for $A_{TM}$

bool willAccept(string program, string input)

Program $P$ design specification:

If $P$ accepts its input, then $P$ does not accept its input.
If $P$ does not accept its input, then $P$ accepts its input.

// Program P

int main() {
    string input = getInput();

    if (willAccept(me, input)) {
        reject();
    } else {
        accept();
    }
}

Crazy as it seems, that’s something we can actually do!
There is a decider $D$ for $A_{TM}$

We can write programs that use $D$ as a helper method

Program $P$ accepts its input if and only if program $P$ does not accept its input

Contradiction!

$A_{TM} \in \mathbb{R}$

First, let's have our program get its own source code. (We know this is possible! We saw how to do it in class.)

```cpp
bool willAccept(string program, string input)
{
    string input = getInput();
    string me = mySource();

    if (willAccept(me, input)) {
        reject();
    } else {
        accept();
    }
}
```

Program $P$ design specification:
- If $P$ accepts its input, then $P$ does not accept its input.
- If $P$ does not accept its input, then $P$ accepts its input.
$A_{TM} \in \mathbb{R}$

There is a decider $D$ for $A_{TM}$

We can write programs that use $D$ as a helper method

Program $P$ accepts its input if and only if program $P$ does not accept its input

Contradiction!

// Program P

```cpp
int main() {
    string input = getInput();
    string me = mySource();

    if (willAccept(me, input)) {
    } else {
    }
}
```

Next, let's call this magic `willAccept` method to ask whether we (program P) are going to accept our input.

A TM $\in \mathbb{R}$

Next, let's call this magic `willAccept` method to ask whether we (program P) are going to accept our input.

Contradiction!
There is a decider $D$ for $A_{TM}$.

We can write programs that use $D$ as a helper method.

Program $P$ accepts its input if and only if $P$ does not accept its input.

Contradiction!

Now, let's look back at our design specification and see what we need to do.

```
bool willAccept(string program, string input)

Program P design specification:
  If $P$ accepts its input, then
    $P$ does not accept its input.
  If $P$ does not accept its input, then
    $P$ accepts its input.

// Program P
int main() {
    string input = getInput();
    string me = mySource();

    if (willAccept(me, input)) {
    } else {
    }
}
```
\( A_{TM} \in \mathbb{R} \)

There is a decider \( D \) for \( A_{TM} \)

We can write programs that use \( D \) as a helper method

Program \( P \) accepts its input if and only if program \( P \) does not accept its input

Contradiction!

\[ \text{bool willAccept(string program, string input)} \]

Program \( P \) design specification:

- If \( P \) accepts its input, then \( P \) does not accept its input.
- If \( P \) does not accept its input, then \( P \) accepts its input.

```c++
// Program P
int main() {
    string input = getInput();
    string me = mySource();
    if (willAccept(me, input)) {
    } else {
    }
}
```

Our specification says that, if this program is supposed to accept its input, then it needs to not accept its input.
A_{TM} \in \mathbb{R}

There is a decider D for A_{TM}

We can write programs that use D as a helper method

Program P accepts its input if and only if program P does not accept its input

Contradiction!

\begin{align*}
\text{bool} \ & \text{willAccept(string program, string input)} \\
\text{Program P design specification:} \\
& \text{If } P \text{ accepts its input, then } P \text{ does not accept its input.} \\
& \text{If } P \text{ does not accept its input, then } P \text{ accepts its input.}
\end{align*}

// Program P

```cpp
int main() {
    string input = getInput();
    string me = mySource();

    if (willAccept(me, input)) {
    } else {
        What's something we can do to not accept our input?
    }
}
```
There is a decider $D$ for $A_{TM}$.

We can write programs that use $D$ as a helper method.

Program $P$ accepts its input if and only if $P$ does not accept its input.

Contradiction!

---

$A_{TM} \in \mathbb{R}$

bool willAccept(string program, string input)

Program $P$ design specification:

- If $P$ accepts its input, then $P$ does not accept its input.
- If $P$ does not accept its input, then $P$ accepts its input.

// Program P

```cpp
int main() {
    string input = getInput();
    string me = mySource();
    if (willAccept(me, input)) {
        reject();
    } else {
    }
}
```
There is a decider $D$ for $A_{TM}$

We can write programs that use $D$ as a helper method

Program $P$ accepts its input if and only if $P$ does not accept its input

Contradiction!

$A_{TM} \in R$

There is a decider $D$ for $A_{TM}$

We can write programs that use $D$ as a helper method

Program $P$ accepts its input if and only if $P$ does not accept its input

Contradiction!

bool willAccept(string program, string input)

Program $P$ design specification:

- If $P$ accepts its input, then $P$ does not accept its input.
- If $P$ does not accept its input, then $P$ accepts its input.

// Program P
int main() {
    string input = getInput();
    string me = mySource();
    if (willAccept(me, input)) {
        reject();
    } else {

    }
}
A_{TM} \in \mathbb{R}

There is a decider $D$ for $A_{TM}$

We can write programs that use $D$ as a helper method

Program $P$ accepts its input if and only if program $P$ does not accept its input

Contradiction!

bool willAccept(string program, string input)

Program $P$ design specification:

- If $P$ accepts its input, then $P$ does not accept its input.
- If $P$ does not accept its input, then $P$ accepts its input.

// Program P

int main() {
  string input = getInput();
  string me = mySource();

  if (willAccept(me, input)) {
    reject();
  } else {
  }
}
\( A_{TM} \in R \)

There is a decider \( D \) for \( A_{TM} \)

We can write programs that use \( D \) as a helper method

Program \( P \) accepts its input if and only if \( program \) \( P \) does not accept its input

Contradiction!

\[
\text{bool willAccept(string program, string input)}
\]

Program \( P \) design specification:

- If \( P \) accepts its input, then \( P \) does not accept its input.
- If \( P \) does not accept its input, then \( P \) accepts its input.

\[
\text{// Program P}
\]

\[
\text{int main() }
\]

\[
\quad \text{string input = getInput();}
\]

\[
\quad \text{string me = mySource();}
\]

\[
\quad \text{if (willAccept(me, input)) }
\]

\[
\quad \quad \text{reject();}
\]

\[
\quad \text{} \quad \text{else }
\]

\[
\quad \quad \text{}
\]

\[
\]

This says that if we aren't supposed to accept the input, then we should accept the input.
There is a decider $D$ for $A_{\text{TM}}$

We can write programs that use $D$ as a helper method

Program $P$ accepts its input if and only if program $P$ does not accept its input

Contradiction!

$A_{\text{TM}} \in \mathbb{R}$

// Program P

```c
int main() {
  string input = getInput();
  string me = mySource();

  if (willAccept(me, input)) {
    reject();
  } else {
    accept();
  }
}
```

A \text{TM} \in \mathbb{R}

So let's go add this line to our program.
There is a decider $D$ for $A_{TM}$

We can write programs that use $D$ as a helper method

Program $P$ accepts its input if and only if $P$ does not accept its input

Contradiction!

$A_{TM} \in \mathbb{R}$

Program $P$ design specification:

- If $P$ accepts its input, then $P$ does not accept its input.
- If $P$ does not accept its input, then $P$ accepts its input.

// Program P

```c
int main() {
    string input = getInput();
    string me = mySource();

    if (willAccept(me, input)) {
        reject();
    } else {
        accept();
    }
}
```

And hey! We're done with this part of the design spec.
\[ A_{TM} \in \mathbb{R} \]

There is a decider \( D \) for \( A_{TM} \).

We can write programs that use \( D \) as a helper method.

Program \( P \) accepts its input if and only if \( P \) does not accept its input.

Contradiction!

\[ \text{bool } \text{willAccept(string program, string input)} \]

Program \( P \) design specification:

- If \( P \) accepts its input, then \( P \) does not accept its input.
- If \( P \) does not accept its input, then \( P \) accepts its input.

```cpp
// Program P
int main() {
    string input = getInput();
    string me = mySource();

    if (willAccept(me, input)) {
        reject();
    } else {
        accept();
    }
}
```

So let's take a quick look over our program \( P \).
There is a decider $D$ for $A_{TM}$.

We can write programs that use $D$ as a helper method.

Program $P$ accepts its input if and only if program $P$ does not accept its input.

Contradiction!

$A_{TM} \in \mathbb{R}$

This is what we said that $P$ was supposed to do. And hey! That's what it does.

// Program P
int main() {
  string input = getInput();
  string me = mySource();

  if (willAccept(me, input)) {
    reject();
  } else {
    accept();
  }
}
\( A_{\text{TM}} \in \mathbb{R} \)

There is a decider \( D \) for \( A_{\text{TM}} \)

We can write programs that use \( D \) as a helper method

Program \( P \) accepts its input if and only if program \( P \) does not accept its input

\( A_{\text{TM}} \in \mathbb{R} \)

There is a decider \( D \) for \( A_{\text{TM}} \)

We can write programs that use \( D \) as a helper method

Program \( P \) accepts its input if and only if program \( P \) does not accept its input

\( \text{bool willAccept(string program, string input)} \)

Program \( P \) design specification:

✓ If \( P \) accepts its input, then \( P \) does not accept its input.
✓ If \( P \) does not accept its input, then \( P \) accepts its input.

// Program P

```c
int main() {
    string input = getInput();
    string me = mySource();

    if (willAccept(me, input)) {
        reject();
    } else {
        accept();
    }
}
```

The whole point of this exercise was to get a contradiction.
Program $P$ design specification:
- If $P$ accepts its input, then $P$ does not accept its input.
- If $P$ does not accept its input, then $P$ accepts its input.

// Program $P$
```java
int main() {
    string input = getInput();
    string me = mySource();
    if (willAccept(me, input)) {
        reject();
    } else {
        accept();
    }
}
```
And, indeed, that’s what we’ve done! There’s a contradiction here because $P$ accepts if and only if it doesn’t accept.
\( A_{TM} \in R \)

There is a decider \( D \) for \( A_{TM} \)

We can write programs that use \( D \) as a helper method

Program \( P \) accepts its input if and only if \( P \) does not accept its input

Contradiction!

bool willAccept(string program, string input)

Program \( P \) design specification:

- If \( P \) accepts its input, then \( P \) does not accept its input.
- If \( P \) does not accept its input, then \( P \) accepts its input.

// Program P
int main() {
    string input = getInput();
    string me = mySource();

    if (willAccept(me, input)) {
        reject();
    } else {
        accept();
    }
}

So if you trace through the implications here...
\( A_{TM} \in \mathbb{R} \)

There is a decider \( D \) for \( A_{TM} \)

We can write programs that use \( D \) as a helper method

Program \( P \) accepts its input if and only if \( P \) does not accept its input

Contradiction!

\[ \text{bool willAccept(string program, string input)} \]

Program \( P \) design specification:

- If \( P \) accepts its input, then \( P \) does not accept its input.
- If \( P \) does not accept its input, then \( P \) accepts its input.

// Program P

```cpp
int main() {
    string input = getInput();
    string me = mySource();
    if (willAccept(me, input)) {
        reject();
    } else {
        accept();
    }
}
```

So if you trace through the implications here...
\( A_{TM} \in \mathbb{R} \)

There is a decider \( D \) for \( A_{TM} \)

We can write programs that use \( D \) as a helper method

Program \( P \) accepts its input if and only if program \( P \) does not accept its input

Contradiction!

\[
\text{bool willAccept(string program, string input)}
\]

Program \( P \) design specification:

- If \( P \) accepts its input, then
  - \( P \) does not accept its input.
- If \( P \) does not accept its input, then
  - \( P \) accepts its input.

// Program P

```python
int main() {
    string input = getInput();
    string me = mySource();
    if (willAccept(me, input)) {
        reject();
    } else {
        accept();
    }
}
```

So if you trace through the implications here...
\( A_{\text{TM}} \in \mathbb{R} \)

There is a decider \( D \) for \( A_{\text{TM}} \)

We can write programs that use \( D \) as a helper method

Program \( P \) accepts its input if and only if program \( P \) does not accept its input

Contradiction!

\[
\text{bool \ willAccept(string program, string input)}
\]

Program \( P \) design specification:

- If \( P \) accepts its input, then \( P \) does not accept its input.
- If \( P \) does not accept its input, then \( P \) accepts its input.

// Program P

```cpp
int main() {
    string input = getInput();
    string me = mySource();

    if (willAccept(me, input)) {
        reject();
    } else {
        accept();
    }
}
```

So if you trace through the implications here...
There is a decider $D$ for $A_{TM}$.

We can write programs that use $D$ as a helper method.

Program $P$ accepts its input if and only if program $P$ does not accept its input.

Contradiction!

---

**bool** willAccept(string program, string input)

Program $P$ design specification:

- If $P$ accepts its input, then $P$ does not accept its input.
- If $P$ does not accept its input, then $P$ accepts its input.

---

// Program P

```cpp
int main() {
    string input = getInput();
    string me = mySource();
    
    if (willAccept(me, input)) {
        reject();
    } else {
        accept();
    }
}
```

You can see that the starting assumption that $A_{TM}$ is decidable leads to a contradiction - we're done!
What does this program do?

```c++
bool willAccept(string program, string input) {
    /* ... some implementation ... */
}

int main() {
    string me = mySource();
    string input = getInput();

    if (willAccept(me, input)) {
        reject();
    } else {
        accept();
    }
}
```

Here's that initial lecture slide again.

Try running this program on any input. What happens if

... this program accepts its input? It rejects the input!

... this program doesn't accept its input? It accepts the input!
What does this program do?

```cpp
bool willAccept(string program, string input) {
    /* ... some implementation ... */
}

int main() {
    string me = mySource();
    string input = getInput();

    if (willAccept(me, input)) {
        reject();
    } else {
        accept();
    }
}
```

Try running this program on any input.

What happens if...

... this program accepts its input?
It rejects the input!

... this program doesn't accept its input?
It accepts the input!

Take a look at it more closely.
What does this program do?

```c++
bool willAccept(string program, string input) {
    /* ... some implementation ... */
}

int main() {
    string me = mySource();
    string input = getInput();

    if (willAccept(me, input)) {
        reject();
    } else {
        accept();
    }
}
```

Try running this program on any input. What happens if...

- ...this program accepts its input? It rejects the input!
- ...this program doesn't accept its input? It accepts the input!

Recognize this code? Now you know where it comes from!
What does this program do?

```c
bool willAccept(string program, string input) {
    /* … some implementation … */
}

int main() {
    string me = mySource();
    string input = getInput();

    if (willAccept(me, input)) {
        reject();
    } else {
        accept();
    }
}
```

We created it to get these contradictions.

Try running this program on any input.

What happens if

... this program accepts its input?
    It rejects the input!

... this program doesn't accept its input?
    It accepts the input!
A_{TM} \in \mathbb{R}

There is a decider $D$ for $A_{TM}$

We can write programs that use $D$ as a helper method

Program $P$ accepts its input if and only if $P$ does not accept its input

Contradiction!

$\text{bool} \ \text{willAccept}(\text{string} \ \text{program}, \ \text{string} \ \text{input})$

Program $P$ design specification:

- If $P$ accepts its input, then $P$ does not accept its input.
- If $P$ does not accept its input, then $P$ accepts its input.

// Program $P$

```c
int main() {
    string input = getInput();
    string me = mySource();

    if (willAccept(me, input)) {
        reject();
    } else {
        accept();
    }
}
```

This might seem like a lot – and in many ways it is.
The key idea here is what's given over there on the left column.
There is a decider $D$ for $A_{TM}$

We can write programs that use $D$ as a helper method

Program $P$ accepts its input if and only if program $P$ does not accept its input

Contradiction!

A_{TM} \in \mathbb{R}

bool willAccept(string program, string input)

Program $P$ design specification:
- If $P$ accepts its input, then $P$ does not accept its input.
- If $P$ does not accept its input, then $P$ accepts its input.

// Program P

int main() {
    string input = getInput();
    string me = mySource();

    if (willAccept(me, input)) {
        reject();
    } else {
        accept();
    }
}

This progression comes up in all the self-reference proofs we've done this quarter.
There is a decider \( D \) for \( A_{TM} \):

- We can write programs that use \( D \) as a helper method.
- Program \( P \) accepts its input if and only if \( P \) does not accept its input.

Contradiction!

\( A_{TM} \in \mathbb{R} \)

Program \( P \) design specification:
- If \( P \) accepts its input, then \( P \) does not accept its input.
- If \( P \) does not accept its input, then \( P \) accepts its input.

We’ll do another example of this in a little bit.

```cpp
// Program P
int main() {
    string input = getInput();
    string me = mySource();

    if (willAccept(me, input)) {
        reject();
    } else {
        accept();
    }
}
```
There is a decider $D$ for $A_{TM}$

We can write programs that use $D$ as a helper method

Program $P$ accepts its input if and only if program $P$ does not accept its input

Contradiction!

$A_{TM} \in \mathbb{R}$

\[
\begin{align*}
M \xrightarrow{W} & \text{Decider } D \\ & \text{for } A_{TM} \\
& \text{willAccept} \\
\end{align*}
\]

\[\text{Yes, } M \text{ accepts } w. \]

\[\text{No, } M \text{ does not accept } w. \]

\[
\text{bool willAccept(string program, string input)}
\]

Program $P$ design specification:

\[\begin{align*}
& \text{If } P \text{ accepts its input, then} \\
& \hspace{1cm} P \text{ does not accept its input.} \\
& \text{If } P \text{ does not accept its input, then} \\
& \hspace{1cm} P \text{ accepts its input.}
\end{align*}\]

// Program P

```cpp
int main() {
    string input = getInput();
    string me = mySource();
    if (willAccept(me, input)) {
        reject();
    } else {
        accept();
    }
}
```

Before we move on, though, I thought I'd take a minute to talk about a few common questions we get.
A_{TM} \in \mathbb{R}

There is a decider $D$ for $A_{TM}$

We can write programs that use $D$ as a helper method

Program $P$ accepts its input if and only if $P$ does not accept its input

Contradiction!

bool willAccept(string program, string input)

Program $P$ design specification:

- If $P$ accepts its input, then $P$ does not accept its input.
- If $P$ does not accept its input, then $P$ accepts its input.

// Program P

```cpp
int main() {
    string input = getInput();
    string me = mySource();

    if (willAccept(me, input)) {
        reject();
    } else {
        accept();
    }
}
```

First, let's jump back to this part of the program $P$ that we wrote.
A_{TM} \in \mathbb{R}

There is a decider $D$ for $A_{TM}$

We can write programs that use $D$ as a helper method

Program $P$ accepts its input if and only if program $P$ does not accept its input

Contradiction!

bool willAccept(string program, string input)

Program $P$ design specification:

- If $P$ accepts its input, then $P$ does not accept its input.
- If $P$ does not accept its input, then $P$ accepts its input.

// Program $P$

int main() {
    string input = getInput();
    string me = mySource();

    if (willAccept(me, input)) {
        reject();
    } else {
        accept();
    }
}
Program $P$ design specification:

- If $P$ accepts its input, then $P$ does not accept its input.
- If $P$ does not accept its input, then $P$ accepts its input.

Here, the specific way we ended up doing that was by having program $P$ reject its input.
There is a decider $D$ for $A_{TM}$

We can write programs that use $D$ as a helper method

Program $P$ accepts its input if and only if program $P$ does not accept its input

Contradiction!

$$A_{TM} \in R$$

We can write programs that use $D$ as a helper method

Program $P$ design specification:

- If $P$ accepts its input, then $P$ does not accept its input.
- If $P$ does not accept its input, then $P$ accepts its input.

// Program $P$

```cpp
int main() {
    string input = getInput();
    string me = mySource();

    if (willAccept(me, input)) {
        reject();
    } else {
        accept();
    }
}
```

I mentioned that there were other things we could do here as well.
A_{TM} \in \mathbb{R}

There is a decider \( D \) for \( A_{TM} \)

We can write programs that use \( D \) as a helper method

Program \( P \) accepts its input if and only if \( P \) does not accept its input

Contradiction!

bool willAccept(string program, string input)

Program \( P \) design specification:

- If \( P \) accepts its input, then \( P \) does not accept its input.
- If \( P \) does not accept its input, then \( P \) accepts its input.

// Program P

int main() {
    string input = getInput();
    string me = mySource();

    if (willAccept(me, input)) {
        while (true) {
        }
    } else {
        accept();
    }
}
There is a decider $D$ for $A_{TM}$

We can write programs that use $D$ as a helper method

Program $P$ accepts its input if and only if $P$ does not accept its input.

Contradiction!
A_{TM} \in \mathbb{R}

There is a decider $D$ for $A_{TM}$

We can write programs that use $D$ as a helper method

Program $P$ accepts its input if and only if $P$ does not accept its input

Contradiction!

// Program P
int main() {
    string input = getInput();
    string me = mySource();

    if (willAccept(me, input)) {
        while (true) { }
    } else {
        accept();
    }
}

A lot of people ask us whether this is allowed, since we were assuming we had a decider and deciders can't loop.
A_{TM} \in \mathbb{R}

There is a decider $D$ for $A_{TM}$

We can write programs that use $D$ as a helper method

Program $P$ accepts its input if and only if program $P$ does not accept its input

Contradiction!

bool willAccept(string program, string input)

Program $P$ design specification:
- If $P$ accepts its input, then $P$ does not accept its input.
- If $P$ does not accept its input, then $P$ accepts its input.

// Program P

int main() {
    string input = getInput();
    string me = mySource();
    if (willAccept(me, input)) {
        while (true) { }
    } else {
        accept();
    }
}
There is a decider \( D \) for \( A_{TM} \)

We can write programs that use \( D \) as a helper method

Program \( P \) accepts its input if and only if \( program \) \( P \) does not accept its input

Contradiction!

\[
A_{TM} \in \mathbb{R}
\]

\[
\text{bool willAccept(string program, string input)}
\]

Program \( P \) design specification:
- If \( P \) accepts its input, then \( P \) does not accept its input.
- If \( P \) does not accept its input, then \( P \) accepts its input.

// Program \( P \)

```cpp
int main() {
    string input = getInput();
    string me = mySource();

    if (willAccept(me, input)) {
        while (true) {}  // infinite loop
    } else {
        accept();
    }
}
```

There are two different programs here.
There is a decider $D$ for $A_{TM}$

We can write programs that use $D$ as a helper method

Program $P$ accepts its input if and only if program $P$ does not accept its input

Contradiction!

bool willAccept(string program, string input)

Program $P$ design specification:

- If $P$ accepts its input, then $P$ does not accept its input.
- If $P$ does not accept its input, then $P$ accepts its input.

// Program P

int main() {
  string input = getInput();
  string me = mySource();

  if (willAccept(me, input)) {
    while (true) {
    }
  } else {
    accept();
  }
}

First, there's this decider $D$. $D$ is a decider, so it's required to halt on all inputs.
There is a decider $D$ for $A_{TM}$

We can write programs that use $D$ as a helper method

Program $P$ accepts its input if and only if program $P$ does not accept its input

Contradiction!

$A_{TM} \in R$

// Program P
int main() {
    string input = getInput();
    string me = mySource();
    if (willAccept(me, input)) {
        while (true) {
        }
    } else {
        accept();
    }
}
There is a decider $D$ for $A_{TM}$

We can write programs that use $D$ as a helper method

Program $P$ accepts its input if and only if program $P$ does not accept its input

Contradiction!

$A_{TM} \in \mathbb{R}$

bool willAccept(string program, string input)

Program $P$ design specification:

If $P$ accepts its input, then $P$ does not accept its input.

If $P$ does not accept its input, then $P$ accepts its input.

// Program P

int main() {
    string input = getInput();
    string me = mySource();

    if (willAccept(me, input)) {
        while (true) { }
    } else {
        accept();
    }
}
There is a decider $D$ for $A_{TM}$

We can write programs that use $D$ as a helper method

Program $P$ accepts its input if and only if $P$ does not accept its input

Contradiction!

$A_{TM} \in R$

bool willAccept(string program, string input)

Program $P$ design specification:
- If $P$ accepts its input, then $P$ does not accept its input.
- If $P$ does not accept its input, then $P$ accepts its input.

The decider is always required to halt, but the program $P$ is not.
There is a decider $D$ for $A_{\text{TM}}$.

We can write programs that use $D$ as a helper method.

Program $P$ accepts its input if and only if $P$ does not accept its input.

Contradiction!
\( A_{\text{TM}} \in \mathbb{R} \)

There is a decider \( D \) for \( A_{\text{TM}} \)

We can write programs that use \( D \) as a helper method

Program \( P \) accepts its input if and only if program \( P \) does not accept its input

Contradiction!

\[ \text{bool willAccept(string program, string input)} \]

Program \( P \) design specification:

✓ If \( P \) accepts its input, then
  ✓ \( P \) does not accept its input.

✓ If \( P \) does not accept its input, then
  ✓ \( P \) accepts its input.

// Program P

```cpp
int main() {
    string input = getInput();
    string me = mySource();

    if (willAccept(me, input)) {
        reject();
    } else {
        accept();
    }
}
```

Much better!
There is a decider $D$ for $A_{\text{TM}}$.

We can write programs that use $D$ as a helper method.

Program $P$ accepts its input if and only if $P$ does not accept its input.

Contradiction!

// Program P
int main() {
    string input = getInput();
    string me = mySource();

    if (willAccept(me, input)) {
        reject();
    } else {
        accept();
    }
}
There is a decider $D$ for $A_{TM}$

We can write programs that use $D$ as a helper method

Program $P$ accepts its input if and only if program $P$ does not accept its input

Contradiction!
There is a decider $D$ for $A_{TM}$

We can write programs that use $D$ as a helper method

Program $P$ accepts its input if and only if $P$ does not accept its input

Contradiction!

\[
A_{TM} \in \mathbb{R}
\]

\[
\text{bool } \text{willAccept}(\text{string program, string input})
\]

Program $P$ design specification:
- If $P$ accepts its input, then $P$ does not accept its input.
- If $P$ does not accept its input, then $P$ accepts its input.

```c
// Program P
int main() {
    string input = getInput();
    string me = mySource();
    if (willAccept(me, input)) {
        reject();
    } else {
        accept();
    }
}
```

... and ask what happens if we take these two lines...
There is a decider $D$ for $A_{TM}$.

We can write programs that use $D$ as a helper method.

Program $P$ accepts its input if and only if $P$ does not accept its input.

Contradiction!

```
// Program P

int main() {
    string input = getInput();
    string me = mySource();

    if (willAccept(me, input)) {
        accept();
    } else {
        reject();
    }
}
```

... and swap them like this.
There is a decider $D$ for $A_{TM}$

We can write programs that use $D$ as a helper method

Program $P$ accepts its input if and only if $P$ does not accept its input

Contradiction!

$A_{TM} \in \mathbb{R}$

There is a decider $D$ for $A_{TM}$

We can write programs that use $D$ as a helper method

Program $P$ accepts its input if and only if $P$ does not accept its input

Contradiction!

bool willAccept(string program, string input)

Program $P$ design specification:

✓ If $P$ accepts its input, then $P$ does not accept its input.
✓ If $P$ does not accept its input, then $P$ accepts its input.

// Program P
int main() {
    string input = getInput();
    string me = mySource();

    if (willAccept(me, input)) {
        accept();
    } else {
        reject();
    }
}

Usually, people ask whether we could have just done this and ended up proving that $A_{TM} \in \mathbb{R}$. 
We can write programs that use $D$ as a helper method.

Program $P$ accepts its input if and only if program $P$ does not accept its input.

Contradiction!
There is a decider $D$ for $A_{TM}$

We can write programs that use $D$ as a helper method

Program $P$ accepts its input if and only if program $P$ does not accept its input

Contradiction!

\[ A_{TM} \in \mathbb{R} \]

bool willAccept(string program, string input)

Program $P$ design specification:

- If $P$ accepts its input, then $P$ does not accept its input.
- If $P$ does not accept its input, then $P$ accepts its input.

// Program $P$

```c++
int main() {
    string input = getInput();
    string me = mySource();
    if (willAccept(me, input)) {
        accept();
    } else {
        reject();
    }
}
```

Notice that this program $P$ doesn’t have the behavior given over here.
There is a decider $D$ for $A_{TM}$

We can write programs that use $D$ as a helper method

Program $P$ accepts its input if and only if program $P$ does accept its input

Contradiction!

$A_{TM} \in R$

bool willAccept(string program, string input)

Program $P$ design specification:

- If $P$ accepts its input, then $P$ does not accept its input.
- If $P$ does not accept its input, then $P$ accepts its input.

// Program P
int main() {
    string input = getInput();
    string me = mySource();

    if (willAccept(me, input)) {
        accept();
    } else {
        reject();
    }
}
There is a decider $D$ for $A_{TM}$

We can write programs that use $D$ as a helper method

Program $P$ accepts its input if and only if program $P$ does accept its input

Contradiction!

$A_{TM} \in \mathbb{R}$

Program $P$ design specification:

- If $P$ accepts its input, then $P$ does not accept its input.
- If $P$ does not accept its input, then $P$ accepts its input.

Notice that this is a true statement.

```cpp
bool willAccept(string program, string input)
```

```cpp
// Program P
int main() {
    string input = getInput();
    string me = mySource();

    if (willAccept(me, input)) {
        accept();
    } else {
        reject();
    }
}
```
A_{TM} ∈ R

There is a decider D for A_{TM}

We can write programs that use D as a helper method

Program P accepts its input if and only if program P does accept its input

Contradiction!

math}

\begin{align*}
\text{Decider } D & \text{ for } A_{TM} \\
\text{Yes, } M \text{ accepts } w. & \\
\text{No, } M \text{ does not accept } w. \\
\end{align*}

bool willAccept(string program, string input)

Program P design specification:

- If P accepts its input, then
  - P does not accept its input.
- If P does not accept its input, then
  - P accepts its input.

// Program P
int main() {
    string input = getInput();
    string me = mySource();

    if (willAccept(me, input)) {
        accept();
    } else {
        reject();
    }
}
There is a decider $D$ for $A_{TM}$

We can write programs that use $D$ as a helper method

Program $P$ accepts its input if and only if program $P$ does accept its input

// Program P

```cpp
int main() {
    string input = getInput();
    string me = mySource();

    if (willAccept(me, input)) {
        accept();
    } else {
        reject();
    }
}
```

Instead, we've shown that we end up at a true statement.
There is a decider $D$ for $A_{TM}$

We can write programs that use $D$ as a helper method

Program $P$ accepts its input if and only if program $P$ does accept its input

However, take a minute to look at the giant implication given here.
There is a decider $D$ for $A_{TM}$

We can write programs that use $D$ as a helper method

Program $P$ accepts its input if and only if $P$ does accept its input.

// Program P

```cpp
int main() {
    string input = getInput();
    string me = mySource();
    if (willAccept(me, input)) {
        accept();
    } else {
        reject();
    }
}
```

Overall, this shows that $A_{TM} \in R \rightarrow T$
There is a decider $D$ for $A_{TM}$

We can write programs that use $D$ as a helper method

Program $P$ accepts its input if and only if $P$ does accept its input

Does this statement say anything about whether $A_{TM}$ is decidable?
There is a decider $D$ for $A_{TM}$

We can write programs that use $D$ as a helper method

Program $P$ design specification:

- If $P$ accepts its input, then $P$ does not accept its input.
- If $P$ does not accept its input, then $P$ accepts its input.

```c
// Program P
int main() {
    string input = getInput();
    string me = mySource();
    if (willAccept(me, input)) {
        accept();
    } else {
        reject();
    }
}
```

A_{TM} \in \mathbb{R} \rightarrow \top

Yes, $M$ accepts $w$.

No, $M$ does not accept $w$.

// Program P

```c
bool willAccept(string program, string input)
```
There is a decider $D$ for $A_{TM}$

We can write programs that use $D$ as a helper method

Program $P$ accepts its input if and only if program $P$ does accept its input

![Diagram](image)

```cpp
// Program P
int main() {
    string input = getInput();
    string me = mySource();

    if (willAccept(me, input)) {
        accept();
    } else {
        reject();
    }
}
```

We have no way of knowing whether $A_{TM} \in R$ or not just by looking at this statement.

$A_{TM} \in R \implies \top$
There is a decider $D$ for $A_{TM}$.

We can write programs that use $D$ as a helper method.

Program $P$ accepts its input if and only if $P$ does not accept its input.

The fact that we didn’t get a contradiction doesn’t mean that $A_{TM}$ is decidable.

$A_{TM} \in \mathbb{R} \rightarrow \top$
There is a decider $D$ for $A_{TM}$

We can write programs that use $D$ as a helper method.

Program $P$ accepts its input if and only if program $P$ does accept its input.

Just so we don’t get confused, let’s reset everything back to how it used to be.
There is a decider $D$ for $A_{TM}$.

We can write programs that use $D$ as a helper method.

Program $P$ accepts its input if and only if $P$ does not accept its input.

Contradiction!

\[
A_{TM} \in \mathbb{R}
\]

**bool** willAccept(string program, string input)

Program $P$ design specification:

- If $P$ accepts its input, then $P$ does not accept its input.
- If $P$ does not accept its input, then $P$ accepts its input.

// Program P

```cpp
int main() {
    string input = getInput();
    string me = mySource();

    if (willAccept(me, input)) {
        reject();
    } else {
        accept();
    }
}
```
There is a decider $D$ for $A_{TM}$

We can write programs that use $D$ as a helper method

Program $P$ accepts its input if and only if $P$ does not accept its input

Contradiction!

$A_{TM} \in R$

bool willAccept(string program, string input)

Program $P$ design specification:

- If $P$ accepts its input, then $P$ does not accept its input.
- If $P$ does not accept its input, then $P$ accepts its input.

// Program P

int main() {
    string input = getInput();
    string me = mySource();

    if (willAccept(me, input)) {
        reject();
    } else {
        accept();
    }
}

Take a look at the general structure of how we got here. Then, let's go do another example.
Do you remember the secure voting problem from lecture?
M is a secure voting machine
if and only if
\[ \mathcal{L}(M) = \{ w \in \{r, d\}^* \mid w \text{ has more } r's \text{ than } d's \} \]

We said that a TM $M$ is a secure voting machine if it obeys the above rule.
M is a secure voting machine
if and only if

\[ L(M) = \{ w \in \{r, d\}^* \mid w \text{ has more } r\text{'s than } d\text{'s } \} \]

That's kind of a lot to take in at once.
M is a secure voting machine if and only if
\[ \mathcal{L}(M) = \{ w \in \{r, d\}^* \mid w \text{ has more } r \text{'s than } d \text{'s} \} \]

Remember – the language of a TM is the set of all the strings it accepts.
M is a secure voting machine if and only if

\[ \mathcal{L}(M) = \{ w \in \{r, d\}^* | w \text{ has more } r\text{'s than } d\text{'s } \} \]

So really this statement means that M accepts every string with more r's than d's and nothing else.
Our goal was to show that it's not possible to build a program that can tell whether an arbitrary TM is a secure voting machine.

\[ \mathcal{L}(M) = \{ w \in \{r, d\}^* \mid w \text{ has more } r's \text{ than } d's \} \]
M is a secure voting machine
if and only if
\[ \mathcal{L}(M) = \{ w \in \{r, d\}^* \mid w \text{ has more } r \text{'s than } d \text{'s } \}\]

Notice that our goal was not to show that you can't build a secure voting machine.
M is a secure voting machine

if and only if

\[ \mathcal{L}(M) = \{ w \in \{r, d\}^* \mid w \text{ has more } r\text{'s than } d\text{'s} \} \]

It's absolutely possible to do that.

```c
int main() {
    string input = getInput();
    if (countRs(input) > countDs(input)) {
        accept();
    } else {
        reject();
    }
}
```
M is a secure voting machine if and only if

\[ \mathcal{L}(M) = \{ w \in \{r, d\}^* \mid w \text{ has more } r\text{'s than } d\text{'s} \} \]

The hard part is being able to tell whether an arbitrary program is a secure voting machine.

```c
int main() {
    string input = getInput();
    if (countRs(input) > countDs(input)) {
        accept();
    } else {
        reject();
    }
}
```
Here's a program where no one knows whether it's a secure voting machine.

```cpp
int main() {
    string input = getInput();

    int n = countRs(input);
    while (n > 1) {
        if (n % 2 == 0) n = n / 2;
        else n = 3*n + 1;
    }

    if (countRs(input) > countDs(input)) {
        accept();
    } else {
        reject();
    }
}
```

M is a secure voting machine if and only if

\[ L(M) = \{ w \in \{r, d\}^* | w \text{ has more } r's \text{ than } d's \} \]
M is a secure voting machine if and only if

\[ \mathcal{L}(M) = \{ w \in \{r, d\}* \mid w \text{ has more } r\text{'s than } d\text{'s } \} \]

You can see this because no one knows whether this part will always terminate.

```c
int main() {
    string input = getInput();
    int n = countRs(input);
    while (n > 1) {
        if (n % 2 == 0) n = n / 2;
        else n = 3*n + 1;
    }
    if (countRs(input) > countDs(input)) {
        accept();
    } else {
        reject();
    }
}
```
M is a secure voting machine if and only if

\[ \mathcal{L}(M) = \{ w \in \{r, d\}^* | w \text{ has more } r\text{'s than } d\text{'s } \} \]

It's entirely possible that this goes into an infinite loop on some input – we're honestly not sure!

```c
int main() {
    string input = getInput();
    
    int n = countRs(input);
    while (n > 1) {
        if (n % 2 == 0) n = n / 2;
        else n = 3*n + 1;
    }

    if (countRs(input) > countDs(input)) {
        accept();
    } else {
        reject();
    }
}
```
M is a secure voting machine
if and only if
\[ L(M) = \{ w \in \{r, d\}^* \mid w \text{ has more } r\text{'s than } d\text{'s} \} \]

So, to recap:
Building a secure voting machine isn’t hard. Checking whether an arbitrary program is a secure voting machine is really hard.

```c
int main() {
    string input = getInput();
    int n = countRs(input);
    while (n > 1) {
        if (n % 2 == 0) n = n / 2;
        else n = 3*n + 1;
    }
    if (countRs(input) > countDs(input)) {
        accept();
    } else {
        reject();
    }
}
```
Our goal is to show that the secure voting problem – the problem of checking whether a program is a secure voting machine – is undecidable.
Following our pattern from before, we’ll assume that the secure voting problem is decidable.
We're ultimately trying to get some kind of contradiction here.
The secure voting problem is decidable.

As before, we'll take it one step at a time.

Contradiction!
First, since we’re assuming that the secure voting problem is decidable, we’re assuming that there’s a decider for it.

Contradiction!
The secure voting problem is decidable. There is a decider $D$ for the secure voting problem.

So what does that look like?

Contradiction!
A decider for the secure voting problem will take in some TM $M$, which is the machine we want to specifically check.
The secure voting problem is decidable.

There is a decider $D$ for the secure voting problem.

The decider will then accept if $M$ is a secure voting machine and reject otherwise.

Contradiction!
The secure voting problem is decidable.

There is a decider $D$ for the secure voting problem.

We can write programs that use $D$ as a helper method.

Following our pattern from before, we’ll then say that we can use this decider as a subroutine in other TMs.

Contradiction!
The secure voting problem is decidable.

There is a decider $D$ for the secure voting problem.

We can write programs that use $D$ as a helper method.

Contradiction!

In software, that decider $D$ might look something like what's given above.

**bool isSecure(string program)**

Yes, $M$ is a secure voting machine.

No, $M$ is not a secure voting machine.
The secure voting problem is decidable.

There is a decider $D$ for the secure voting problem.

We can write programs that use $D$ as a helper method.

Contradiction!

Yes, $M$ is a secure voting machine.

No, $M$ is not a secure voting machine.

```
bool isSecure(string program)
```

Here, isSecure is just another name for the decider $D$, but with a more descriptive name.
The secure voting problem is decidable.

There is a decider \( D \) for the secure voting problem.

We can write programs that use \( D \) as a helper method.

Contradiction!

\begin{itemize}
  \item \textit{Decider} \( D \) for the secure voting problem
  \item \textit{isSecure} \( M \)
  \item \textbf{bool} \textit{isSecure}(\textit{string} \textit{program})
  \item Its argument (\textit{program}) is just a more descriptive name for the TM (\textit{program}) given as input.
\end{itemize}

Yes, \( M \) is a secure voting machine.

No, \( M \) is not a secure voting machine.
The secure voting problem is decidable.

There is a decider $D$ for the secure voting problem.

We can write programs that use $D$ as a helper method.

**Contradiction!**

This was the point in the previous proof where we started to write a design spec for some self-referential program $P$. Yes, $M$ is a secure voting machine. 

No, $M$ is not a secure voting machine.

```cpp
def isSecure(program: str) -> bool:
    # Implementation...
```
Previously, we wrote $P$ to get this contradiction:

"$P$ accepts if and only if it doesn't accept."

The secure voting problem is decidable.

There is a decider $D$ for the secure voting problem.

We can write programs that use $D$ as a helper method.

Contradiction!

```cpp
bool isSecure(string program)
```

Previously, we wrote $P$ to get this contradiction:

"$P$ accepts if and only if it doesn't accept."
The secure voting problem is decidable.

There is a decider $D$ for the secure voting problem.

We can write programs that use $D$ as a helper method.

Contradiction!

That was a great contradiction to get when we had a decider that would tell us whether a program would accept a given input.

```
bool isSecure(string program)
```

```
Decider $D$ for the secure voting problem

isSecure

program

M

Yes, $M$ is a secure voting machine.

No, $M$ is not a secure voting machine.
```
The problem here is that our decider doesn’t do that. Instead, it tells us whether a program is a secure voting machine.

Contradiction!

There is a decider $D$ for the secure voting problem.

We can write programs that use $D$ as a helper method.

The secure voting problem is decidable.

Decider $D$ for the secure voting problem

$M$ is a secure voting machine.

$M$ is not a secure voting machine.

`bool isSecure(string program)`

The problem here is that our decider doesn’t do that. Instead, it tells us whether a program is a secure voting machine.
Following the maxim of “do what you can with what you have where you are,” we’ll try to set up a contradiction concerning whether a program is or is not a voting machine.

The secure voting problem is decidable.

There is a decider $D$ for the secure voting problem.

We can write programs that use $D$ as a helper method.

Contradiction!
Specifically, we're going to build a program \( P \) that is a secure voting machine if and only if it's not a secure voting machine.

The secure voting problem is decidable.

Contradiction!
Generally speaking, you'll try to set up a contradiction where the program has the property given by the decider if and only if it doesn't have the property given by the decider.

The secure voting problem is decidable.

There is a decider $D$ for the secure voting problem

We can write programs that use $D$ as a helper method

Program $P$ is secure if and only if program $P$ is not secure.

Contradiction!
The secure voting problem is decidable.

There is a decider $D$ for the secure voting problem.

We can write programs that use $D$ as a helper method.

Program $P$ is secure if and only if $P$ is not secure.

Contradiction!

Generally speaking, you’ll try to set up a contradiction where the program has the property given by the decider if and only if it doesn’t have the property given by the decider.

Pay attention to that other guy! That’s really, really good advice!

Yes, $M$ is a secure voting machine.

No, $M$ is not a secure voting machine.

`bool isSecure(string program)`
The secure voting problem is decidable.

There is a decider $D$ for the secure voting problem

We can write programs that use $D$ as a helper method

Program $P$ is secure if and only if program $P$ is not secure.

Contradiction!

So now we have to figure out how to write this program $P$. Yes, $M$ is a secure voting machine.

No, $M$ is not a secure voting machine.
As before, let’s start by writing out a design specification for what it’s supposed to do.

The secure voting problem is decidable.

There is a decider $D$ for the secure voting problem.

We can write programs that use $D$ as a helper method.

Program $P$ design specification:

As before, let’s start by writing out a design specification for what it’s supposed to do.
The secure voting problem is decidable.

There is a decider $D$ for the secure voting problem

We can write programs that use $D$ as a helper method

Program $P$ design specification:

If $P$ is a secure voting machine, then $P$ is not a secure voting machine.

This first part takes care of the first half of the biconditional.
The secure voting problem is decidable.

There is a decider $D$ for the secure voting problem

We can write programs that use $D$ as a helper method

Program $P$ design specification:
If $P$ is a secure voting machine, then
$P$ is not a secure voting machine.
If $P$ is not a secure voting machine, then
$P$ is a secure voting machine.

This second part takes care of the other direction.

Contradiction!
The secure voting problem is decidable.

There is a decider $D$ for the secure voting problem.

We can write programs that use $D$ as a helper method.

Program $P$ design specification:
- If $P$ is a secure voting machine, then $P$ is not a secure voting machine.
- If $P$ is not a secure voting machine, then $P$ is a secure voting machine.

At this point, we have written out a spec for what we want $P$ to do. All that’s left to do now is to code it up!
The secure voting problem is decidable.

There is a decider $D$ for the secure voting problem.

We can write programs that use $D$ as a helper method.

Program $P$ design specification:

- If $P$ is a secure voting machine, then $P$ is not a secure voting machine.
- If $P$ is not a secure voting machine, then $P$ is a secure voting machine.

In lecture, we wrote one particular program that met these requirements. For the sake of simplicity, I'm going to write a different one here. Don't worry! It works just fine.
The secure voting problem is decidable.

There is a decider $D$ for the secure voting problem.

We can write programs that use $D$ as a helper method.

Program $P$ design specification:
- If $P$ is a secure voting machine, then $P$ is not a secure voting machine.
- If $P$ is not a secure voting machine, then $P$ is a secure voting machine.

Our program starts off in main().
The secure voting problem is decidable.

There is a decider $D$ for the secure voting problem.

We can write programs that use $D$ as a helper method.

Program $P$ design specification:
If $P$ is a secure voting machine, then $P$ is not a secure voting machine.
If $P$ is not a secure voting machine, then $P$ is a secure voting machine.

Contradiction!

We can write programs that use $D$ as a helper method.

Program $P$ design specification:
If $P$ is a secure voting machine, then $P$ is not a secure voting machine.
If $P$ is not a secure voting machine, then $P$ is a secure voting machine.

// Program $P$
int main() {

Ultimately, we need to figure out if we’re a secure voting machine or not.
}

bool isSecure(string program)
The secure voting problem is decidable.

There is a decider \( D \) for the secure voting problem.

We can write programs that use \( D \) as a helper method.

Program \( P \) design specification:

- If \( P \) is a secure voting machine, then \( P \) is not a secure voting machine.
- If \( P \) is not a secure voting machine, then \( P \) is a secure voting machine.

The best tool we have for that is some kind of self-reference trick.
The secure voting problem is decidable.

There is a decider $D$ for the secure voting problem

We can write programs that use $D$ as a helper method

Program $P$ design specification:
- If $P$ is a secure voting machine, then $P$ is not a secure voting machine.
- If $P$ is not a secure voting machine, then $P$ is a secure voting machine.

As before, we'll use the fact that we have this decider lying around to make $P$ figure out what exactly it does.
The secure voting problem is decidable.

There is a decider \( D \) for the secure voting problem.

We can write programs that use \( D \) as a helper method.

Program \( P \) design specification:
- If \( P \) is a secure voting machine, then \( P \) is not a secure voting machine.
- If \( P \) is not a secure voting machine, then \( P \) is a secure voting machine.

Specifically, let’s have program \( P \) ask what it’s going to do.
The secure voting problem is decidable.

There is a decider $D$ for the secure voting problem.

We can write programs that use $D$ as a helper method.

Program $P$ design specification:
- If $P$ is a secure voting machine, then $P$ is not a secure voting machine.
- If $P$ is not a secure voting machine, then $P$ is a secure voting machine.

Let's take it one step at a time.

```cpp
// Program P
int main() {
    string me = mySource();
    if (isSecure(me)) {
    } else {
    }
}
```
The secure voting problem is decidable.

There is a decider $D$ for the secure voting problem.

We can write programs that use $D$ as a helper method.

Program $P$ design specification:
- If $P$ is a secure voting machine, then $P$ is not a secure voting machine.
- If $P$ is not a secure voting machine, then $P$ is a secure voting machine.

// Program P
int main() {
    string me = mySource();
    if (isSecure(me)) {
    } else {
        // Oddly enough, let's look at the second requirement first. Why? I ask: why not?
    }
}
The secure voting problem is decidable.

There is a decider $D$ for the secure voting problem.

We can write programs that use $D$ as a helper method.

Program $P$ design specification:
- If $P$ is a secure voting machine, then $P$ is not a secure voting machine.
- If $P$ is not a secure voting machine, then $P$ is a secure voting machine.

This requirement says that if the program is supposed to not be a secure voting machine, then it needs to be a secure voting machine.

```c++
// Program P
int main() {
    string me = mySource();
    if (isSecure(me)) {
    } else {
    }
}
```
The secure voting problem is decidable.

There is a decider $D$ for the secure voting problem

We can write programs that use $D$ as a helper method

Program $P$ design specification:
- If $P$ is a secure voting machine, then $P$ is not a secure voting machine.
- If $P$ is not a secure voting machine, then $P$ is a secure voting machine.

We can write programs that use $D$ as a helper method.

Program $P$ is secure if and only if program $P$ is not secure.

Contradiction!
The secure voting problem is decidable.

There is a decider $D$ for the secure voting problem.

We can write programs that use $D$ as a helper method.

Program $P$ design specification:
- If $P$ is a secure voting machine, then $P$ is not a secure voting machine.
- If $P$ is not a secure voting machine, then $P$ is a secure voting machine.

In this specific case, we're suppose to make $P$ be a secure voting machine.

```c
// Program P
int main() {
    string me = mySource();
    if (isSecure(me)) {
        accept();
    } else {
        if (countRs(input) > countDs(input)) accept();
        else reject();
    }
}
```
The secure voting problem is decidable.

There is a decider $D$ for the secure voting problem

We can write programs that use $D$ as a helper method

Program $P$ design specification:

If $P$ is a secure voting machine, then $P$ is not a secure voting machine.

If $P$ is not a secure voting machine, then $P$ is a secure voting machine.

That means we need to make $P$ accept all strings with more r's than d's and not accept anything else.

```java
// Program P
int main() {
    string me = mySource();
    if (isSecure(me)) {
    } else {
    }
}
```
The secure voting problem is decidable.

There is a decider $D$ for the secure voting problem.

We can write programs that use $D$ as a helper method.

Program $P$ design specification:

If $P$ is a secure voting machine, then $P$ is not a secure voting machine.

If $P$ is not a secure voting machine, then $P$ is a secure voting machine.

The good news is that, a while back, we already saw how to do that!

// Program $P$
int main() {
    string me = mySource();
    if (isSecure(me)) {
        accept();
    } else {
        if (countRs(input) > countDs(input)) accept();
        else reject();
    }
}
The secure voting problem is decidable.

There is a decider $D$ for the secure voting problem.

We can write programs that use $D$ as a helper method.

Program $P$ design specification:

If $P$ is a secure voting machine, then $P$ is not a secure voting machine.

If $P$ is not a secure voting machine, then $P$ is a secure voting machine.

The code looks something like this.

```c++
// Program P
int main() {
    string input = getInput();
    string me = mySource();
    if (isSecure(me)) {
        accept();
    } else {
        if (countRs(input) > countDs(input)) accept();
        else reject();
    }
}
```
The secure voting problem is decidable. There is a decider $D$ for the secure voting problem.

We can write programs that use $D$ as a helper method.

Program $P$ design specification:
- If $P$ is a secure voting machine, then $P$ is not a secure voting machine.
- If $P$ is not a secure voting machine, then $P$ is a secure voting machine.

Just to confirm that this works – notice that if the input has more r’s than d’s, we accept it, and otherwise we reject.
The secure voting problem is decidable.

There is a decider $D$ for the secure voting problem

We can write programs that use $D$ as a helper method

Program $P$ design specification:
If $P$ is a secure voting machine, then
$P$ is not a secure voting machine.
✓ If $P$ is not a secure voting machine, then
$P$ is a secure voting machine.

Program $P$ is secure if and only if program $P$ is not secure.

Contradiction!

// Program P
int main() {
    string input = getInput();
    string me = mySource();
    if (isSecure(me)) {
        accept();
    } else {
        if (countRs(input) > countDs(input)) accept();
        else reject();
    }
}

Okay! So that's one of two requirements down.
The secure voting problem is decidable.

There is a decider $D$ for the secure voting problem

We can write programs that use $D$ as a helper method

Program $P$ design specification:
If $P$ is a secure voting machine, then
$P$ is not a secure voting machine.
✓ If $P$ is not a secure voting machine, then
$P$ is a secure voting machine.

Let’s move on to the other one.
The secure voting problem is decidable.

There is a decider $D$ for the secure voting problem

We can write programs that use $D$ as a helper method

Program $P$ design specification:

- If $P$ is a secure voting machine, then $P$ is not a secure voting machine.
- If $P$ is not a secure voting machine, then $P$ is a secure voting machine.

We can write programs that use $D$ as a helper method

Program $P$ is secure if and only if program $P$ is not secure.

Contradiction!
The secure voting problem is decidable.

There is a decider $D$ for the secure voting problem

We can write programs that use $D$ as a helper method

Program $P$ design specification:

- If $P$ is a secure voting machine, then $P$ is not a secure voting machine.
- If $P$ is not a secure voting machine, then $P$ is a secure voting machine.

We can write programs that use $D$ as a helper method

Program $P$ is secure if and only if program $P$ is not secure.

Contradiction!
The secure voting problem is decidable.

There is a decider $D$ for the secure voting problem.

We can write programs that use $D$ as a helper method.

Program $P$ design specification:

If $P$ is a secure voting machine, then
$P$ is not a secure voting machine.

If $P$ is not a secure voting machine, then
$P$ is a secure voting machine.

We can literally do anything we want except accepting all strings with more r's than d's and not accepting anything else.
The secure voting problem is decidable.

There is a decider $D$ for the secure voting problem.

We can write programs that use $D$ as a helper method.

Program $P$ design specification:

If $P$ is a secure voting machine, then $P$ is not a secure voting machine.

If $P$ is not a secure voting machine, then $P$ is a secure voting machine.

Program $P$:

```cpp
int main() {
    string input = getInput();
    string me = mySource();

    if (isSecure(me)) {
        accept();
    } else {
        if (countRs(input) > countDs(input)) accept();
        else reject();
    }
}
```

Among the many things we can do that falls into the "literally anything else" camp would be to just accept everything.
The secure voting problem is decidable.

There is a decider $D$ for the secure voting problem.

We can write programs that use $D$ as a helper method.

Program $P$ design specification:
- If $P$ is a secure voting machine, then $P$ is not a secure voting machine.
- If $P$ is not a secure voting machine, then $P$ is a secure voting machine.

Notice that in this case, $P$ is not a secure voting machine: it accepts everything, including a ton of strings it's not supposed to.

### Program $P$

```cpp
// Program P
int main() {
    string input = getInput();
    string me = mySource();
    if (isSecure(me)) {
        accept();
    } else {
        if (countRs(input) > countDs(input)) accept();
        else reject();
    }
}
```
The secure voting problem is decidable.

There is a decider $D$ for the secure voting problem

We can write programs that use $D$ as a helper method

Program $P$ design specification:

$\checkmark$ If $P$ is a secure voting machine, then $P$ is not a secure voting machine.

$\checkmark$ If $P$ is not a secure voting machine, then $P$ is a secure voting machine.

So we're done with this part of the design!
The secure voting problem is decidable.

There is a decider $D$ for the secure voting problem.

We can write programs that use $D$ as a helper method.

Program $P$ design specification:

- If $P$ is a secure voting machine, then $P$ is not a secure voting machine.
- If $P$ is not a secure voting machine, then $P$ is a secure voting machine.

We can write programs that use $D$ as a helper method.

Program $P$ is secure if and only if program $P$ is not secure.

Contradiction!

Putting it all together, take a look at what we accomplished. This program is a secure voting machine if and only if it isn't a secure voting machine!
The secure voting problem is decidable.

There is a decider $D$ for the secure voting problem

We can write programs that use $D$ as a helper method

Program $P$ design specification:

- \( P \) is a secure voting machine, then \( P \) is not a secure voting machine.
- \( P \) is not a secure voting machine, then \( P \) is a secure voting machine.

That gives us the contradiction that we needed to get.

\[
\text{bool isSecure(string program)}
\]

// Program P

```c++
int main() {
    string input = getInput();
    string me = mySource();

    if (isSecure(me)) {
        accept();
    } else {
        if (countRs(input) > countDs(input)) accept();
        else reject();
    }
}
```
The secure voting problem is decidable.

There is a decider $D$ for the secure voting problem.

We can write programs that use $D$ as a helper method.

Program $P$ design specification:

- If $P$ is a secure voting machine, then $P$ is not a secure voting machine.
- If $P$ is not a secure voting machine, then $P$ is a secure voting machine.

We're done! We've shown that starting with the assumption that the secure voting problem is decidable, we reach a contradiction.
Let’s take a minute to review the general process that we followed to get these results to work.
Let's take a minute to review the general process that we followed to get these results to work.

That other guy is going to tell you a general pattern to follow. You might want to take notes.
Let's suppose that you want to prove that some language about TMs is undecidable.
Start off by assuming it's decidable.
The goal is to get a contradiction. The problem in question is decidable.
The problem in question is decidable.

Contradiction!

To get there...
The problem in question is decidable.

There is a decider $D$ for that problem.

...the first step is to suppose that you have a decider for the language in question.

Contradiction!
The problem in question is decidable.

There is a decider $D$ for that problem.

Decider $D$ for this problem

It's often a good idea to draw a picture showing what that decider looks like.

Contradiction!
Think about what the inputs to the decider are going to look like. That depends on the language.

The problem in question is decidable.

There is a decider $D$ for that problem.
In the cases we’re exploring in this class, there will always be at least one input that’s a TM of some sort.

The problem in question is decidable.

There is a decider $D$ for that problem.

Contradiction!
The problem in question is decidable.

There is a decider $D$ for that problem.

Next, think about what the decider is going to tell you about those inputs. That depends on the problem at hand.

Contradiction!
For example, if your language is the set of TMs that have some property $X$, then the decider will tell you whether the TM has property $X$. 

The problem in question is decidable.

There is a decider $D$ for that problem.

Yes, $M$ has property $X$.

No, $M$ doesn't have property $X$. 

Contradiction!
The problem in question is decidable.

There is a decider $D$ for that problem.

We can write programs that use $D$ as a helper method.

The next step is to think about how to use that decider as a subroutine in some program.
Think about what the decider would look like as a method in some high-level programming language.

The problem in question is decidable.

There is a decider $D$ for that problem.

We can write programs that use $D$ as a helper method.

Contradiction!
You already know what inputs it's going to take and what it says, so try to come up with a nice, descriptive name for the method.
In this case, since our decider says whether the program has some property X, a good name would be something like hasPropertyX.

The problem in question is decidable.

There is a decider D for that problem.

We can write programs that use D as a helper method.

Contradiction!
The problem in question is decidable.

There is a decider $D$ for that problem.

We can write programs that use $D$ as a helper method.

Contradiction!

It doesn't hurt to label the decider $D$ to show what parts of the decider correspond with the method.

```cpp
bool hasPropertyX(string program)
```
The next step is to build a self-referential program that gives you some sort of contradiction.

The problem in question is decidable.

There is a decider $D$ for that problem.

We can write programs that use $D$ as a helper method.

Contradiction!
You're going to want to get a contradiction by building a program that has some property $X$ if and only if it doesn't have some property $X$.

The problem in question is decidable.

There is a decider $D$ for that problem.

We can write programs that use $D$ as a helper method.

Program $P$ has property $X$ if and only if $P$ doesn't have property $X$.

Contradiction!
The problem in question is decidable.

There is a decider $D$ for that problem.

We can write programs that use $D$ as a helper method.

Program $P$ has property $X$ if and only if $P$ doesn't have property $X$.

Contradiction!

### Diagram:

- **Decider $D$ for this problem**
  - **Yes, $M$ has property $X$.**
  - **No, $M$ doesn't have property $X$.**

### Code:

```cpp
bool hasPropertyX(string program) {
    // Implementation
}
```

Now, you have to figure out how to write program $P$. 
We recommend writing out a design specification for the program that you're going to write.

The problem in question is decidable. There is a decider $D$ for that problem. We can write programs that use $D$ as a helper method.

Program $P$ has property $X$ if and only if $P$ doesn't have property $X$.

Contradiction!
The problem in question is decidable.

There is a decider $D$ for that problem.

We can write programs that use $D$ as a helper method.

Program $P$ has property $X$ if and only if $P$ doesn't have property $X$.

Contradiction!

**bool** `hasPropertyX(string program)`

Program $P$ design specification:

- If $P$ has property $X$, then $P$ does not have property $X$.
- If $P$ does not have property $X$, then $P$ has property $X$.

You can fill out that spec by reasoning about both directions of the implication.
The problem in question is decidable.

There is a decider $D$ for that problem.

We can write programs that use $D$ as a helper method.

Program $P$ design specification:

If $P$ has property $X$, then $P$ does not have property $X$.
If $P$ does not have property $X$, then $P$ has property $X$.

Finally, you have to go and write a program that gives you a contradiction.
The problem in question is decidable.

There is a decider $D$ for that problem.

We can write programs that use $D$ as a helper method.

Program $P$ has property $X$ if and only if $P$ doesn't have property $X$.

Contradiction!

// Program $P$

```cpp
int main() {
    string input = getInput();
    string me = mySource();

    if (hasPropertyX(me)) {
        // do something so you don't have property $X$.
    } else {
        // Do something so you do have property $X$.
    }
}
```

If you follow the design spec, you'll likely get something like this. Filling in the blanks takes some creativity.
The problem in question is decidable.

There is a decider \( D \) for that problem.

We can write programs that use \( D \) as a helper method.

Program \( P \) has property \( X \) if and only if \( P \) doesn't have property \( X \)

Contradiction!
Hope this helps!

Please feel free to ask questions if you have them.
Did you find this useful? If so, let us know! We can go and make more guides like these.