

Problems for Week One

Problem One: A Quick Algebra Review

In the first week of CS103, we'll be doing a few proofs that will require some algebraic manipulations and reasoning about equalities. To make sure that this material is fresh in your memory for then, take a few minutes to work through these problems.

- i. Simplify this product of polynomials: $(2n + 1)(3n - 2)$.
- ii. Factor this polynomial: $n^2 + 4n + 3$.
- iii. Expand the expressions $(2k)^2$ and $(2k + 1)^2$. (These particular expressions will come up on Wednesday and Friday of this week.)
- iv. Write the following as a single fraction: $\frac{2m+1}{n} + \frac{2p+1}{q}$.

Problem Two: Natural Numbers, Integers, and Real Numbers

Our first lecture on mathematical proofs will explore properties of the natural numbers, the integers, and the real numbers. Although there's a reasonable chance that you've learned about these types of numbers in one of your high school math classes, we suspect that you probably haven't actually needed to use these numbers mathematically outside of, say, CS106A/B. To prepare for Wednesday and Friday's lectures on these subjects, take a few minutes to work through the following problems.

- i. Let n be a natural number and consider the quantity $4n^2 + 4n + 1$. Is this quantity...
 - guaranteed to be a natural number, or might it depend on n ?
 - guaranteed to be an integer, or might it depend on n ?
 - guaranteed to be a real number, or might it depend on n ?
- ii. Let a , b , and c be natural numbers and consider the quantity $ab - c$. Is this quantity...
 - guaranteed to be a natural number, or might it depend on a , b , and c ?
 - guaranteed to be an integer, or might it depend on a , b , and c ?
 - guaranteed to be a real number, or might it depend on a , b , and c ?
- iii. Let a , b , and c be integers, where $c \neq 0$, and consider the quantity ab/c . Is this quantity...
 - guaranteed to be a natural number, or might it depend on a , b , and c ?
 - guaranteed to be an integer, or might it depend on a , b , and c ?
 - guaranteed to be a real number, or might it depend on a , b , and c ?

Problem Three: Set Theory Symbol Review

In our first lecture, we introduced a bunch of new symbols in the context of set theory. As a reference, here's a list of all the symbols we encountered:

$$\in \notin \emptyset \mathbb{N} \mathbb{Z} \mathbb{R} \cup \cap - \Delta \subseteq \not\subseteq \wp \cdot | \aleph_0$$

What do each of the above symbols mean? Give an example of how each might be used.

Problem Four: Set Theory Concept Review

Below are some questions about sets and set theory. Answer each question, briefly justifying your answer.

- i. Is $1 \in \{0, 1, 2, 3\}$?
- ii. Is $1 \subseteq \{0, 1, 2, 3\}$?
- iii. Is $1 \in \{0, 1, 2, 3\} \cap \{0, 2, 4, 6\}$?
- iv. Is $1 \in \{0, 1, 2, 3\} \cup \{0, 2, 4, 6\}$?
- v. Is $1 \in \{0, 1, 2, 3\} - \{0, 2, 4, 6\}$?
- vi. Is $1 \in \{0, 2, 4, 6\} - \{0, 1, 2, 3\}$?
- vii. Is $1 \in \{0, 1, 2, 3\} \Delta \{0, 2, 4, 6\}$?
- viii. Is $\emptyset \in \{0, 1, 2, 3\}$?
- ix. Is $\emptyset \subseteq \{0, 1, 2, 3\}$?
- x. Is $\wp(\{1\}) = \{1\}$?
- xi. Is $|\{0, 1, 2, 3\}| = |\{0, 10, 20, 30\}|$?
- xii. Is $\emptyset = \{\emptyset\}$?
- xiii. Is $\emptyset = \{\{\emptyset\}\}$?
- xiv. Is $\{\emptyset, \{\emptyset\}\} = \{\emptyset\}$?

Problem Five: Set-Builder Notation

Consider the following set, which we'll call S :

$$S = \{ n \in \mathbb{N} \mid n \text{ is odd} \}.$$

Below is a mathematical argument. Is it correct?

“Let’s pick the number $n = 137$. We know that n is odd because 137 doesn’t cleanly divide by two (also, its last digit is a 7). We also know that 137 is a natural number, since it’s a whole number and isn’t negative. So that means that $n \in \mathbb{N}$ and n is odd. Therefore, we see that $S = \{137\}$.”

Problem Six: Set-Builder Notation, Part II

For the purposes of this problem, we're going to invent some new notation. If T is a triangle, we'll let $S_1(T)$, $S_2(T)$, and $S_3(T)$ be the lengths of the three sides of T written in non-decreasing order. That is, for any triangle T , we have $S_1(T) \leq S_2(T) \leq S_3(T)$. For example, if T is a triangle with side lengths 3, 4, and 5, then $S_1(T) = 3$, $S_2(T) = 4$, and $S_3(T) = 5$. If T is a triangle with side lengths 6, 6, and 4, then $S_1(T) = 4$ and $S_2(T) = S_3(T) = 6$.

Consider these three sets:

$$A = \{ T \mid T \text{ is a triangle and } S_1(T) = S_2(T) \}$$

$$B = \{ T \mid T \text{ is a triangle and } S_2(T) = S_3(T) \}$$

$$C = \{ T \mid T \text{ is a triangle} \}$$

This question explores how you might combine these sets together to represent particular sets of triangles.

- i. Draw a Venn diagram showing how these three sets overlap with one another.
- ii. Using just the standard set operations (union, intersection, difference, etc.) and the three sets given above, write an expression that means “the set of all equilateral triangles.” (An equilateral triangle is one where all three sides are the same.) As a hint, think about what region of the Venn diagram corresponds to this set.
- iii. Using just the standard set operations (union, intersection, difference, etc.) and the three sets given above, write an expression that means “the set of all isosceles triangles.” (An isosceles triangle is one where at least two of the three sides are equal.)
- iv. Using just the standard set operations (union, intersection, difference, etc.) and the three sets given above, write an expression that means “the set of all isosceles triangles that aren't equilateral.”
- v. Using just the standard set operations (union, intersection, difference, etc.) and the three sets given above, write an expression that means “the set of all scalene triangles.” (A scalene triangle is one where all three sides have different lengths.)

Problem Seven: Exploring Set Theory

Each of the following questions will ask you to find some sets that meet various properties. You'll probably need to play around with them a bit before you'll find something that works, but that's okay! Try things out and see what you come up with.

- i. Find sets A and B where $A \not\subseteq B$, but $A \subseteq B$.
- ii. Find sets A and B where $A \in B$, but $A \not\subseteq B$.
- iii. Find sets A and B where $A \in B$ and $A \subseteq B$.
- iv. Find a set A where $A \in \wp(A)$.
- v. Find a set A where $A \subseteq \wp(A)$.
- vi. Find a set A where $|A| = 2$ and $|A \cup \wp(A)| = 6$.
- vii. Find a set A where $|A| = 2$ and $|A \cup \wp(A)| = 4$. (*Yes, this is possible!*)

Problem Eight: Sets, Subsets, and Cardinality

Here's a problem that builds up to a pretty nifty result about sets and set theory.

Let's begin by defining a new set $A = \{ n \in \mathbb{N} \mid n < 137 \}$.

i. How many elements are in the set A ? What are they?

With that set A in mind, let's define the set $E = \{ S \subseteq A \mid |S| \text{ is even} \}$.

ii. Describe the set E in plain English, then list off a couple examples of elements of E . (Zero is considered even.)

Finally, let's have $O = \{ S \subseteq A \mid |S| \text{ is odd} \}$.

iii. Describe the set O in plain English, then list off a couple examples of elements of O . (Zero is not considered odd.)

iv. Show that $|E| = |O|$ by finding a way of pairing off all the elements of E with elements of O . There are a lot of ways to do this – try to see if you can find a simple rule for doing so.

As a hint for part (iv) of this problem, when you're faced with the task of working with large, complicated sets, sometimes it's easier to consider smaller cases of the same problem and to see if you can find a pattern. For example, it might be easier to spot a pattern if you replace A with the set $\{ n \in \mathbb{N} \mid n < 3 \}$.