CS103A
Winter 2020

Handout 01S

Solutions for Week One

Problem One: A Quick Algebra Review

In the first week of CS103, we'll be doing a few proofs that will require some algebraic manipulations and reasoning about equalities. To make sure that this material is fresh in your memory for then, take a few minutes to work through these problems.

i. Simplify this product of polynomials: $(2n + 1)(3n - 2)$.

This simplifies to

$$\begin{aligned} & 6n^2 - 4n + 3n - 2 \\ = \; & 6n^2 - n - 2. \end{aligned}$$

ii. Factor this polynomial: $n^2 + 4n + 3$.

This polynomial can be factored into (n + 1)(n + 3).

iii. Expand the expressions $(2k)^2$ and $(2k + 1)^2$. (These particular expressions will come up on Wednesday and Friday of this week.)

The quantity $(2k)^2$ expands to $4k^2$. The expression $(2k + 1)^2$ expands to $4k^2 + 4k + 1$.

iv. Write the following as a single fraction: $\frac{2m+1}{n} + \frac{2p+1}{q}$.

Multiplying the numerator and denominator of the first fraction by q and the second fraction by n gives us common denominators:

$$\frac{2m+1}{n} + \frac{2p+1}{q} = \frac{2mq+q}{nq} + \frac{2pn+n}{nq}.$$

We can then add the numerators because they share a common denominator:

$$\frac{2mq+q}{nq} + \frac{2pn+n}{nq} = \frac{2mq+2pn+q+n}{nq}.$$

This gives a final answer of $\frac{2mq+2pn+q+n}{nq}$.

Problem Two: Natural Numbers, Integers, and Real Numbers

i. Let n be a natural number and consider the quantity $4n^2 + 4n + 1$. Is this quantity…

- guaranteed to be a natural number, or might it depend on n?

Yes, this is guaranteed to be a natural number, since the product and sum of any two natural numbers is always a natural number.

- guaranteed to be an integer, or might it depend on *n*?

Yes. Since the expression is guaranteed to be a natural number and all the natural numbers are integers, the expression is always an integer.

- guaranteed to be a real number, or might it depend on *n*?

Yes. Since the expression is guaranteed to be a natural number and all the natural numbers are real numbers, the expression is always a real number.

i. Let a, b, and c be natural numbers and consider the quantity $ab - c$. Is this quantity…

- guaranteed to be a natural number, or might it depend on a, b, and c?

It depends on a, b, and c. The natural numbers are all nonnegative, so if we can find a way to make this expression negative we'll show that it doesn't always come out to an integer. For example, we can pick a = 0, b = 0, and c = 1 to get

$$ab - c = 0 \cdot 0 - 1 = -1.$$

However, this will sometimes be a natural number, say, if we pick a = 0, b = 0, and c = 0, where the whole expression evaluates to 0.

- guaranteed to be an integer, or might it depend on a, b, and c?

Yes, this will always be an integer. All natural numbers are integers, so a, b, and c are all integers, and the sum, product, or difference of any two integers is also an integer.

- guaranteed to be a real number, or might it depend on a, b, and c?

Yes, since it's always an integer and all integers are real numbers, it's always a real number.

i. Let a, b, and c be integers, where $c \neq 0$, and consider the quantity $^{ab}/_{c}$. Is this quantity…

- guaranteed to be a natural number, or might it depend on a, b, and c?

No, this will not necessarily be a natural number. For example, if a = -1, b = 2, and c = 1, then this evaluates to -2, which is not a natural number. However, it might be a natural number; try picking a = 5, b = 5, and c = 5 and this works out to 5.

- guaranteed to be an integer, or might it depend on a, b, and c?

No, this will not always be an integer. The quotient of two integers isn't necessarily an integer. For example, if we pick a = 2, b = 3, and c = 5, then this is $^{6}/_{5}$, which is not an integer. However, the expression might be an integer if we pick a, b, and c strategically; say, if a = 1, b = 1, and c = 1.

- guaranteed to be a real number, or might it depend on a, b, and c?

Yes, this is always a real number. The quotient of two nonzero integers is always a real number.

Problem Three: Set Theory Symbol Review

In our first lecture, we introduced a bunch of new symbols in the context of set theory. As a reference, here's a list of all the symbols we encountered:

$$\in \quad \notin \quad \emptyset \quad \mathbb{N} \quad \mathbb{Z} \quad \mathbb{R} \quad \cup \quad \cap \quad - \quad \Delta \quad \subseteq \quad \not\subseteq \quad \wp \quad |\cdot| \quad \aleph_0$$

What do each of the above symbols mean? Give an example of how each might be used.

The $\in$ symbol is the "element-of" symbol. It means that an object belongs to a set. For example, we can say $1 \in \{1, 2, 3\}$.

The $\notin$ symbol is the "not-element-of" symbol. It means that an object doesn't belong to a set. For example, we can say $1 \notin \{2, 3, 4\}$.

The $\emptyset$ symbol denotes the empty set { }, which contains no elements. We might say, for example, that $1 \notin \emptyset$.

The symbols $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$ denote the sets of all natural numbers, all integers, and all real numbers, respectively. For example, we might say $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{R}$.

The $\cup$ and $\cap$ symbols denote set union and set intersection, respectively. For example, we have that $\{1, 2, 3\} \cup \{3, 4, 5\} = \{1, 2, 3, 4, 5\}$ and $\{1, 2, 3\} \cap \{3, 4, 5\} = \{3\}$.

The – and Δ symbols denote set difference and set symmetric difference, respectively. For example, we have $\{1, 2, 3\} - \{3, 4, 5\} = \{1, 2\}$ and $\{1, 2, 3\} \Delta \{3, 4, 5\} = \{1, 2, 4, 5\}$.

The $\subseteq$ and $\not\subseteq$ symbols denote the "is a subset of" and "is not a subset of" relations, respectively. They denote when all the elements of one set belong to a second, or when some element of a set doesn't belong to another set, respectively. For example, $\{1, 2, 3\} \subseteq \{1, 2, 3, 4\}$, but we also have that $\{1, 2, 3, 4\} \not\subseteq \{1, 2, 3\}$.

The $\wp$ symbol denotes the powerset operation, which takes in a set and produces the set of all its subsets. For example, $\wp(\{1, 2\}) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$.

The $|\cdot|$ vertical bars denotes set cardinality, the number of elements in a set. For example, we could say that $|\{11, 222, 3333, 4444\}| = 4$.

Finally, the $\aleph_0$ symbol denotes the cardinality of the set $\mathbb{N}$. Specifically, $|\mathbb{N}| = \aleph_0$.

Problem Four: Set Theory Concept Review

i. Is $1 \in \{0, 1, 2, 3\}$?

Yes, because 1 is an element of the set {0, 1, 2, 3}.

ii. Is $1 \subseteq \{0, 1, 2, 3\}$?

No, 1 is not a subset of {0, 1, 2, 3}, because 1 is not a set.

iii. Is $1 \in \{0, 1, 2, 3\} \cap \{0, 2, 4, 6\}$?

First, a note. Is this supposed to be interpreted as

$$(1 \in \{0, 1, 2, 3\}) \quad \cap \quad \{0, 2, 4, 6\}$$

or as

$$1 \quad \in \quad (\{0, 1, 2, 3\} \cap \{0, 2, 4, 6\})?$$

Let's look at each option, looking at the types involved. For the first possible parenthesization, the statement 1 ∈ {0, 1, 2, 3} evaluates to something akin to a boolean – either 1 is in that set or it isn't. The left-hand side is a set. And that means that we're trying to apply the set intersection operator to a boolean and a set, which doesn't mathematically make sense (you can only intersect two sets).

On the other hand, that second option means "intersect these two sets, then see whether 1 is an element of the result." That actually type-checks, and that's the intended meaning. If you ever have questions about operator precedence, you can often solve them by using checks like these!

To determine this one, let's evaluate the right-hand side. The set {0, 1, 2, 3} ∩ {0, 2, 4, 6} is the intersection of the two sets {0, 1, 2, 3} and {0, 2, 4, 6}. This is the set of all elements in common to the two sets, which is {0, 2}. This set doesn't contain 1, so the answer is "no."

iv. Is $1 \in \{0, 1, 2, 3\} \cup \{0, 2, 4, 6\}$?

The set on the right-hand side is the union of the two sets {0, 1, 2, 3} and {0, 2, 4, 6}, which is the set {0, 1, 2, 3, 4, 6}. This set contains 1, so the answer is "yes."

v. Is $1 \in \{0, 1, 2, 3\} - \{0, 2, 4, 6\}$?

The set on the right-hand side is the difference of the two sets {0, 1, 2, 3} and {0, 2, 4, 6}, which can be formed by starting with the elements in {0, 1, 2, 3} and removing anything in {0, 2, 4, 6}. The resulting set is {1, 3}, and 1 is indeed an element of this set, so the answer is "yes."

vi. Is $1 \in \{0, 2, 4, 6\} - \{0, 1, 2, 3\}$?

The set on the right-hand side is the difference of the two sets {0, 2, 4, 6} and {0, 1, 2, 3}, which can be formed by starting with the elements in {0, 2, 4, 6} and removing anything in {0, 1, 2, 3}. The resulting set is {4, 6}, which doesn't contain 1. Therefore, the answer is "no."

vii. Is $1 \in \{0, 1, 2, 3\} \Delta \{0, 2, 4, 6\}$?

The set on the right-hand side is the symmetric difference of the sets {0, 2, 4, 6} and {0, 1, 2, 3}, which consists of all elements that belong to exactly one of the two indicated sets. This is the set {1, 3, 4, 6}, which does contain 1, so the answer is "yes."

viii. Is $\emptyset \in \{0, 1, 2, 3\}$?

No, the empty set is not an element of {0, 1, 2, 3}. The set {0, 1, 2, 3} contains four elements, namely the numbers zero, one, two, and three, and none of these are the empty set.

ix. Is $\emptyset \subseteq \{0, 1, 2, 3\}$?

Yes, the empty set is a subset of {0, 1, 2, 3}. The empty set is a subset of every set, including the set {0, 1, 2, 3}.

x. Is $\wp(\{1\}) = \{1\}$?

To answer this question, let's determine ℘({1}). This is the power set of {1}, the set of all the subsets of {1}. The set {1} has two subsets – the empty set Ø, which is a subset of every set, and the set {1}, since every set is a subset of itself. There are no other subsets of this set. Therefore, the power set of {1} is {Ø, {1}}, which is not the set {1}, so the answer is "no."

xi. Is $|\{0, 1, 2, 3\}| = |\{0, 10, 20, 30\}|$?

Each of these sets contain exactly four elements, so each set has cardinality four. Therefore, the answer is "yes."

xii. Is $\emptyset = \{\emptyset\}$?

No, these sets are not equal. The set Ø contains no elements at all. On the other hand, the set {Ø} does contain an element, the empty set. This means that one set contains something the other doesn't, so the two sets aren't equal.

xiii. Is $\emptyset = \{\{\emptyset\}\}$?

No, these sets are not equal. The set Ø contains no elements at all. On the other hand, the set {{Ø}} does contain an element, the set {Ø}. This means that one set contains something the other doesn't, so the two sets aren't equal.

xiv. Is {Ø, {Ø}} = {Ø}?

No, these sets are not equal. The set on the left has two elements, Ø and {Ø}, while the set on the right just has one element, Ø.

Problem Five: Set-Builder Notation

Consider the following set, which we'll call S:

$$S = \{ n \in \mathbb{N} \mid n \text{ is odd } \}.$$

Below is a mathematical argument. Is it correct?

> "Let's pick the number $n = 137$. We know that n is odd because 137 doesn't cleanly divide by two (also, its last digit is a 7). We also know that 137 is a natural number, since it's a whole number and isn't negative. So that means that $n \in \mathbb{N}$ and n is odd. Therefore, we see that $S = \{137\}$."

Nope, this argument is incorrect. The argument is totally right that n ∈ **ℕ and that *n* is odd, but that doesn't mean that *S* = {137}. The statement *S* = {137} means "the set *S* contains 137 and nothing else." But this isn't true: for example, the number 103 is also odd and also a natural number, and so 103 ∈ *S*.**

Problem Six: Set-Builder Notation, Part II

For the purposes of this problem, we're going to invent some new notation. If T is a triangle, we'll let $S_1(T)$, $S_2(T)$, and $S_3(T)$ be the lengths of the three sides of T written in non-decreasing order. That is, for any triangle T, we have $S_1(T) \leq S_2(T) \leq S_3(T)$.

Consider these three sets:

$$A = \{ T \mid T \text{ is a triangle and } S_1(T) = S_2(T) \}$$

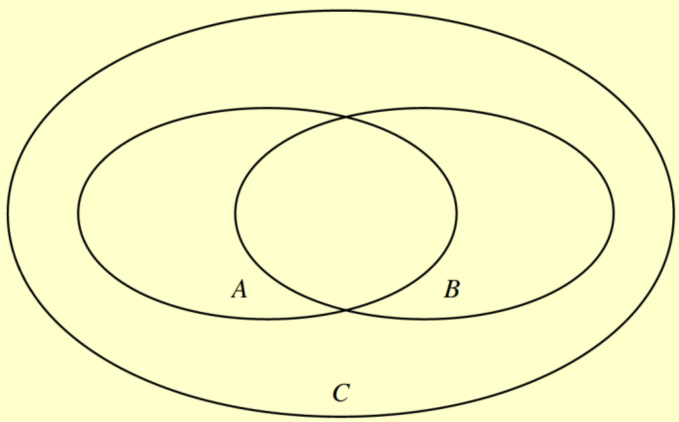
$$B = \{ T \mid T \text{ is a triangle and } S_2(T) = S_3(T) \}$$

$$C = \{ T \mid T \text{ is a triangle} \}$$

This question explores how you might combine these sets together to represent particular sets of triangles.

i. Draw a Venn diagram showing how these three sets overlap with one another.

Here's how these sets relate to one another:



Notice that sets ***A* and *B* are entirely contained within set *C*, and that *A* and *B* overlap one another.**

ii. Using just the standard set operations (union, intersection, difference, etc.) and the three sets given above, write an expression that means "the set of all equilateral triangles." (An equilateral triangle is one where all three sides are the same.) As a hint, think about what region of the Venn diagram corresponds to this set.

An equilateral triangle is one where all three sides are the same. In our notation, that would be a triangle where **$S_1(T)$, $S_2(T)$, and $S_3(T)$ are all equal, or, equivalently, where**

$$S_1(T) = S_2(T) = S_3(T).$$

So which triangles have these properties? Well, that would be the triangles that are in A (since $S_1(T) = S_2(T)$) and also in B (since $S_2(T) = S_3(T)$), and with some thought we can see that every triangle in both A and B would be equilateral. Therefore, we want all the triangles in both A and B, which is the set $A \cap B$.

iii. Using just the standard set operations (union, intersection, difference, etc.) and the three sets given above, write an expression that means "the set of all isosceles triangles." (An isosceles triangle is one where at least two of the three sides are equal.)

An isosceles triangle is one where at least two sides are equal. Since the side lengths are assumed to be in nondecreasing order, that means that if two sides are equal, they're either the shortest two sides or the longest two sides (or both). That means that a triangle will be isosceles precisely if $S_1(T) = S_2(T)$ or $S_2(T) = S_3(T)$. (Since the sides are in sorted order, if we have that $S_1(T) = S_3(T)$, it means that we also have both that $S_1(T) = S_2(T)$ and $S_2(T) = S_3(T)$. Do you see why?)

The set of all triangles with at least one of these properties is therefore $A \cup B$, since any triangle with the first property belongs to the first set and any triangle with the second property belongs to the second set.

iv. Using just the standard set operations (union, intersection, difference, etc.) and the three sets given above, write an expression that means "the set of all isosceles triangles that aren't equilateral."

There are a couple of different ways to write this one out. One observation is that we have expressions for the set of all isosceles triangles and the set of all equilateral triangles, so we could just use set subtraction to express the difference. This would give $(A \cup B) – (A \cap B)$.

Another option would be to notice that if a triangle is isosceles but not equilateral, then exactly one of the statements $S_1(T) = S_2(T)$ and $S_2(T) = S_3(T)$ would be true. We can capture the set of all triangles with exactly one of these two properties as $A \Delta B$, since this is the set of all triangles that has one, but not both, of the properties.

A nifty consequence of this is that we can conclude that $A \Delta B = (A \cup B) – (A \cap B)$. This is a nice little identity to keep in your back pocket and, later on, when we're talking about propositional logic, you'll see something like this used as a way of describing exclusive or.

v. Using just the standard set operations (union, intersection, difference, etc.) and the three sets given above, write an expression that means "the set of all scalene triangles." (A scalene triangle is one where all three sides have different lengths.)

A useful observation here is that a triangle is scalene precisely if it isn't isoceles – if all three sides are different, it means that no two sides are the same, and conversely, if any two of the sides are the same, it means that all three sides can't be different. This means that we could write this one out at $C – (A \cup B)$.

Problem Seven: Exploring Set Theory

i. Find sets A and B where $A \notin B$, but $A \subseteq B$.

There are many answers here. One answer would be $A = \mathbb{N}$ and $B = \mathbb{Z}$. The set $\mathbb{N}$ is not itself an element of $\mathbb{Z}$, since $\mathbb{N}$ is a set and $\mathbb{Z}$ doesn't contain any sets. However, $\mathbb{N} \subseteq \mathbb{Z}$ because every element of $\mathbb{N}$ is also an element of $\mathbb{Z}$.

ii. Find sets A and B where $A \in B$, but $A \nsubseteq B$.

One possible answer would be to pick $A = \mathbb{N}$ and $B = \{\mathbb{N}\}$. Notice that $\mathbb{N} \in \{\mathbb{N}\}$ because the set $\mathbb{N}$ is itself a direct member of $\{\mathbb{N}\}$. However, $\mathbb{N}$ is not a subset of $\{\mathbb{N}\}$. To see this, notice that the set $\mathbb{N}$ contains 0, but the set $\{\mathbb{N}\}$ does not (it only contains one element, $\mathbb{N}$).

iii. Find sets A and B where $A \in B$ and $A \subseteq B$.

One possible answer would be to pick $A = \emptyset$ and $B = \{\emptyset\}$. Notice that $\emptyset \in \{\emptyset\}$ because the set $\emptyset$ is itself a direct member of $\{\emptyset\}$. Also, since $\emptyset$ is a subset of all sets, it's in particular a subset of the set $\{\emptyset\}$.

iv. Find a set A where $A \in \wp(A)$.

Any set works! Since any set is a subset of itself, every set is contained within its power set.

v. Find a set A where $A \subseteq \wp(A)$.

This one's trickier! One simple answer is $\emptyset$. Remember that $\emptyset$ is a subset of all sets, so in particular it must be a subset of $\wp(\emptyset)$.

vi. Find a set A where $|A| = 2$ and $|A \cup \wp(A)| = 6$.

One option is $A = \{137, 42\}$. Here, $|A| = 2$ because A has two elements. The set $\wp(A)$ is the set $\{\emptyset, \{137\}, \{42\}, \{137, 42\}\}$. Therefore, $A \cup \wp(A) = \{137, 42, \emptyset, \{137\}, \{42\}, \{137, 42\}\}$. As a result, $|A \cup \wp(A)| = 6$, because there are six different elements in it.

vii. Find a set A where $|A| = 2$ and $|A \cup \wp(A)| = 4$.

This one is tricky. No matter what set A you pick, you know in advance that $\wp(A)$ is a set containing sets. Also, with a bit of experimentation, you can see that if $|A| = 2$, then $|\wp(A)| = 4$ regardless of what elements are contained in the set A. Therefore, if you want $|A \cup \wp(A)|$ to be 4, you'll have to pick a set containing other sets.

With a bit of trial and error – and based on some of the observations from part (v) of this problem – you might end up trying $A = \{\emptyset, \{\emptyset\}\}$. Here, $\wp(A) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$, so very carefully taking $A \cup \wp(A)$ shows that $A \cup \wp(A) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$, which has cardinality four.

Problem Eight: Sets, Subsets, and Cardinality

Here's a problem that builds up to a pretty nifty result about sets and set theory.

i. Consider the set $A = \{ n \in \mathbb{N} \mid n < 137 \}$. How many elements are in the set A? What are they?

This set contains 137 elements, namely, the natural numbers 0, 1, 2, ..., 136.

ii. Consider the set $E = \{ S \in A \mid |S| \text{ is even} \}$. Describe the set E in plain English, then list off a couple examples of elements of E. (Zero is considered even.)

This is the set of all subsets of A that contain an even number of elements. Stated differently, it's the set of all sets formed from an even number of the natural numbers 0, 1, 2, ..., 136. Some example sets in E include Ø, {1, 136}, {103, 106, 107, 109}, and {0, 1, 2, 3, 4, 5}.

iii. Consider the set $O = \{ S \in A \mid |S| \text{ is odd} \}$. Describe the set O in plain English, then list off a couple examples of elements of O. (Zero is not considered odd.)

This is the set of all subsets of A that contain an odd number of elements. Stated differently, it's the set of all sets formed from an odd number of the natural numbers 0, 1, 2, ..., 136. Some example sets in E include {136}, {103}, {103, 106, 107}, and {41, 42, 43}.

iv. Show that $|E| = |O|$ by finding a way of pairing off all the elements of E with elements of O. There are a lot of ways to do this – try to see if you can find a simple rule for doing so.

There are many ways to pair these elements off in the way that's requested. With a little bit of playing around, you might find that one "natural" way to do this is to pair off every set S with its complement set, the set A – S. For example, we'd pair Ø with {0, 1, 2, ..., 136}, we'd pair {0} with {1, 2, 3, ..., 136}, and we'd pair {0, 2, 4, 6, 8, ..., 136} with {1, 3, 5, 7, ..., 135}. Notice that any set with an even number of elements gets paired up with a set with an odd number of elements – can you explain why?

This idea can be generalized to show that any set S with an odd number of elements has the same number of subsets with an even number of elements as subsets with an odd number of elements. If you take CS109, you'll see how to calculate exactly how many sets of each size there are!