Problems for Week Two

Problem One: Concept Checks

We've covered quite a lot of ground in the past couple of lectures, so let's begin by reviewing some of the major concepts from the past week.

i. What is a universal statement?

ii. How do you prove a universal statement with a direct proof?

iii. What is an existential statement?

iv. How do you prove an existential statement with a direct proof?

v. What an an implication?

vi. What is an antecedent?

vii. What is a consequent?

viii. How do you prove an implication with a direct proof?

ix. What is a lemma?

x. What is a proof by cases?

xi. What is the contrapositive of the statement "if P is true, then Q is true?"

xii. What is the negation of the statement "if P is true, then Q is true?"

xiii. What is the negation of the statement "for all x, $P(x)$ is true?"

xiv. What is the negation of the statement "there is an x where $P(x)$ is true?"

Problem Two: Properties of Integers

In lecture, we proved that if n is an integer, then n is even if and only if n^2 is even. This question explores some other properties about the relationship between n and n^2. Our hope is that this gives you a chance to play around with proofs and proof techniques.

An integer n is called a ***multiple of four*** if n can be written as $4k$ for some integer k. Using terminology analogous to the checkpoint problem on Problem Set One, a number is called congruent to one modulo four if it can be written as $4k + 1$ for some integer k, congruent to two modulo four if it can be written as $4k + 2$ for some integer k, and congruent to three modulo four if it can be written as $4k + 3$ for some integer k. Every integer is either a multiple of four, congruent to one modulo four, congruent to two modulo four, or congruent to three modulo four, and all these options are mutually exclusive.

This question concerns a particular property of numbers modulo four. Specifically, for any integer n, the following statement is true:

$$\text{If } n^2 \text{ is not a multiple of four, then } n \text{ is odd.} \qquad (\bigstar)$$

We'd like to help you build up to a full proof of this result.

i. What is the contrapositive of statement ($\bigstar$)?

ii. Prove statement ($\bigstar$) using a proof by contrapositive.

iii. What is the negation of statement ($\bigstar$)?

iv. Prove statement ($\bigstar$) using a proof by contradiction.

v. Let's suppose that n is an arbitrary integer. Can you say anything about remainder you get when you divide n^2 by four? Play around, see what you find, then formalize your result by writing up a formal mathematical proof.

vi. *(A fun problem to work through if you have some extra time.)* According to the US Census Bureau estimates, the population of the United States on January 1, 2016 was 322,761,807. Prove that there are no integers m and n such that $m^2 + n^2 = 322{,}761{,}807$. As a hint, think about your result from part (v) and consider pulling out a calculator. As a further hint, this would be a great spot to use a proof by contradiction – try assuming that somehow it is possible to find an m and n with certain properties and show that this leads to a contradiction.

Problem Three: Balls in Bins

Suppose that you have twenty-five balls to place into five different bins. Eleven of the balls are red, and the other fourteen are blue. Prove that no matter how the balls are placed into the bins, there must be at least one bin containing at least three red balls. As a hint, when given a problem like this one, it's often a good idea to try out a few concrete examples to see if you spot a pattern.

Problem Four: Pythagorean Triples

A *Pythagorean triple* is a triple of positive natural numbers (a, b, c) such that $a^2 + b^2 = c^2$. For example, $(3, 4, 5)$ is a Pythagorean triple because $3^2 + 4^2 = 9 + 16 = 25 = 5^2$.

i. Prove that if (a, b, c) is a Pythagorean triple, then at least one of a, b, and c is even.

ii. Prove that if (a, b, c) is a Pythagorean triple, then $(a+1, b+1, c+1)$ is *not* a Pythagorean triple.

Problem Five: Proofs on Set Theory

In the middle of class, we'll take a break to prove the following result: if A and B are arbitrary sets, then $\wp(A) \cap \wp(B) \subseteq \wp(A \cap B)$. Once we've done that and you've seen that proof, prove the following statement: if A and B are any sets, then

$$\wp(A) \cap \wp(B) = \wp(A \cap B).$$

To do so, write down what you know so far about how to prove that two sets are equal. Given that, and given what we've already proven, what do you need to show? Try to model your proof along the lines of the one we did as a group: go one step at a time, unpack definitions, clearly articulate where you're going, and give names to the relevant quantities.

Problem Six: A Review of Propositional Logic

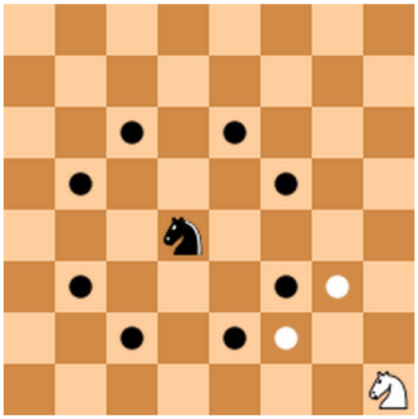
List the seven propositional connectives and give their truth tables. Check your answers by using the online truth table tool.

Problem Seven: Knight's Tours

If you finish everything else, here's a fun little problem to work through.

In chess, the knight piece moves in an L shape by moving two steps in one direction, then one step in a direction perpendicular to the initial direction. To the right is an image from Wikipedia showing a white and black knight piece and where those pieces can move.

A ***knight's tour*** is a sequence of knight moves starting from some position on a chessboard that visits every square on the board exactly once. Many famous mathematicians and computer scientists – including Donald Knuth, one of the most influential computer scientists of all time – have studied the mathematical and computational properties of knight's tours, and now we'd like you to follow in their footsteps!



Prove that there is no knight's tour that starts in the bottom-left corner of the board and ends in the upper-right corner of the board.