Solutions for Week Three

Problem One: Concept Checks

i. There are five propositional connectives besides $\top$ and $\bot$. What are they?

The five other propositional connectives are $\land$, $\lor$, $\to$, $\leftrightarrow$, and $\neg$. If you need a refresher on any of these, check out the Truth Table Tool!

ii. What is the negation of the formula $p \to q$? Repeat this exercise for the four remaining propositional connectives.

The negation of $p \to q$ is $p \land \neg q$. For the remaining connectives:

- $p \lor q$ negates to $\neg p \land \neg q$.
- $p \land q$ negates either to $p \to \neg q$ or $\neg p \lor \neg q$.
- $\neg p$ negates to p.
- $p \leftrightarrow q$ negates to either $\neg p \leftrightarrow q$ or $p \leftrightarrow \neg q$.

iii. In a propositional logic formula, what does each variable represent? In a first-order logic formula, what does each variable represent?

In propositional logic, each variable (and formula) stands for a proposition, something that's either true or false. In a first-order logic formula, each variable stands for an object.

iv. What is the difference between a predicate and a function?

Predicates produce propositions as output, and functions produce objects as output.

v. Can predicates and functions be applied to objects?

Yes! In fact, they can *only* be applied to objects!

vi. Can predicates and functions be applied to propositions?

No, they cannot be applied to propositions.

vii. There's a propositional connective that often pairs with the $\forall$ quantifier. Which is it?

It's $\to$. You should be very careful if you see $\forall$ paired with $\land$!

viii. There's a propositional connective that often pairs with the $\exists$ quantifier. Which is it?

It's $\land$. You should be very careful if you see $\exists$ paired with $\rightarrow$!

ix. In first-order logic, is the equality symbol (=) a predicate, a function, both, or neither?

Equality is a predicate – it takes in two objects and produces a proposition.

Problem Two: Implications are Weird

The "implies" connective $\rightarrow$ is one of the stranger connectives. Below are a series of statements regarding implications. For each statement, confirm that it is indeed true, then briefly explain why.

i. For any propositions P and Q, the following is always true: $(P \rightarrow Q) \lor (Q \rightarrow P)$.

Here's one way to see this. If Q is true, then $P \rightarrow Q$ is true because anything implies a true statement. If Q is false, then $Q \rightarrow P$ is true because false implies anything. (If this is confusing, you should review the truth table for $\rightarrow$!)

ii. More generally, for any propositions P, Q, and R, the following statement is always true: $(P \rightarrow Q) \lor (Q \rightarrow R)$.

This is basically the same argument as before. If Q is true, then $P \rightarrow Q$ is true because anything implies a true statement. If Q is false, then $Q \rightarrow R$ is true because false implies anything.

Why we asked this question: Of all the connectives we've seen, the $\rightarrow$ connective is probably the trickiest. We asked this question to force you to disentangle notions of correlation or causality from the behavior of the $\rightarrow$ connective.

Problem Three: Designing Propositional Formulas

Below is a series of English descriptions of relations among propositional variables. For each description, write a propositional formula that precisely encodes that relation. Then, briefly explain the intuition behind your formula. Try to see if you can come up with the simplest formula possible.

i. For the variables p, q, and r: exactly one of p, q, and r is true.

One option is $(p \land \neg q \land \neg r) \lor (\neg p \land q \land \neg r) \lor (\neg p \land \neg q \land r)$. This essentially lists all possible combinations of how exactly one variable could be true.

ii. For the variables a, b, c, and d: If any of the variables are true, then all the variables that follow it alphabetically in the English alphabet are also true.

One option is $(a \to b) \land (b \to c) \land (c \to d)$. This says that if any variable is true, the one immediately after it must be true. Transitively, this guarantees that if any variable is true, everything after it is true as well.

Why we asked this question: This question was designed to get you thinking about how to use the propositional connectives to express larger and more complicated ideas. The propositional connectives are surprisingly expressive, and we hoped that this question would help you build an intuition behind how they work and how to use them.

Problem Four: True or False?

Below is a list of statements written in first-order logic. For each statement, translate it into English, then decide whether it's true or false.

i. $\forall n \in \mathbb{N}.\ \exists m \in \mathbb{N}.\ n < m$

True. This says "for every natural number, there's a larger natural number."

ii. $\exists n \in \mathbb{N}.\ \forall m \in \mathbb{N}.\ n < m$

False. This says "there is a natural number that's smaller than all natural numbers." No matter what you pick for n, if you pick $m = n$, then you'll have $n \geq m$. Remember that quantifiers can talk about the same object at the same time!

iii. $\forall n \in \mathbb{N}.\ \forall m \in \mathbb{N}.\ (n < m \to \exists p \in \mathbb{N}.\ (n < p \land p < m))$

False. This says "there is a natural number between any two natural numbers." If you pick $n = 0$ and $m = 1$, you cannot find a natural number p where $0 < p$ and $p < 1$.

iv. $\forall n \in \mathbb{R}.\ \forall m \in \mathbb{R}.\ (n < m \to \exists p \in \mathbb{R}.\ (n < p \land p < m))$

True. This says "there is a real number between any two real numbers." Given two different real numbers n and m, the real number $(n + m) / 2$ is between n and m.

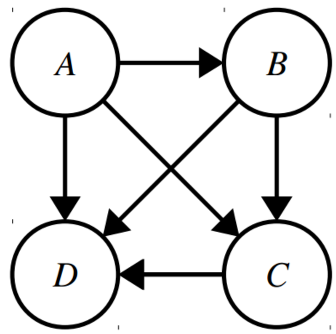
v. $\forall n \in \mathbb{N}. \forall m \in \mathbb{N}. \exists p \in \mathbb{N}. (n = p \cdot m)$

False. This says "for any two natural numbers, the first is a multiple of the second." Try picking $n = 5$ and $m = 3$; there's no choice of p that works.

vi. $\forall n \in \mathbb{R}. \forall m \in \mathbb{R}. \exists p \in \mathbb{R}. (n = p \cdot m)$

False. This says "for any two real numbers, there is a real number you can multiply the second number by to get the first." Try picking $n = 1$ and $m = 0$.

Why we asked this question: This question was designed to help you practice translating statements out of first-order logic. We hoped that you'd get a feel for how to read alternating quantifiers and would then have the follow-up task of reasoning about properties of natural and real numbers.



Problem Five: Interpersonal Dynamics

The diagram to the right represents a set of people named *A*, *B*, *C*, and *D*. If there's an arrow from a person *x* to a person *y*, then person *x* loves person *y*. We'll denote this by writing *Loves*(*x*, *y*). Below is a list of formulas in first-order logic about the above picture. In those formulas, the letter *P* represents the set of all the people. For each formula, determine whether that formula is true or false.

i. $\forall x \in P. \forall y \in P. (Loves(x, y) \lor Loves(y, x))$

This statement is false. Pick *x* and *y* to be *A*. Then *Loves*(*x*, *y*) is false and *Loves*(*y*, *x*) is false. Remember that quantifiers can range over the same objects at the same time!

ii. $\forall x \in P. \forall y \in P. (x \neq y \to Loves(x, y) \lor Loves(y, x))$

This statement is true – given any pair of two people in this diagram, one of them loves the other.

iii. $\forall x \in P. \forall y \in P. (x \neq y \to (Loves(x, y) \leftrightarrow \neg Loves(y, x)))$

This statement is true. Given any pair of two people, exactly one of them loves the other, so either *Loves(x, y)* will be true, or *Loves(y, x)* will be true, but not both. The biconditional in this case will therefore always evaluate to true.

iv. $\exists x \in P. \forall y \in P. (Loves(x, y))$

This statement is false – no one loves everyone, because no one loves themselves.

v. $\exists x \in P. \forall y \in P. (x \neq y \to Loves(x, y))$

This statement is true – pick *x* to be person *A*.

vi. $\forall y \in P. \exists x \in P. (Loves(x, y))$

This statement is false. No one loves person *A*.

vii. $\forall y \in P. \exists x \in P. (x \neq y \land Loves(x, y))$

This statement is still false – no one loves person *A*.

viii. $\exists x \in P. \forall y \in P. (\neg Loves(x, y))$

This statement is true – pick *x* to be person *D*.

Why we asked this question: As we start moving into the realm of binary relations (and later, other discrete structures), we're going to start seeing a lot of definitions given purely in first-order logic. You'll need to be able to look at a first-order statement and some particular object in question, then think about whether the first-order statement is true in that case. These problems were designed to get you thinking about how to read and interpret first-order logic formulas in a case that bears a surprising resemblance to what you'll end up doing for binary relations.

Problem Six: Negating Statements

i. $\exists k.\ (RugbyPlayer(k) \land FootballPlayer(k) \land 49er(k))$

There are many different ways you can negate this. We're going to use the following equivalence:

$$\neg(p \land q \land r) \equiv p \land q \to \neg r$$

Given this, here's one option:

$$\neg\exists k.\ (RugbyPlayer(k) \land FootballPlayer(k) \land 49er(k))$$

$$\forall k.\ \neg(RugbyPlayer(k) \land FootballPlayer(k) \land 49er(k))$$

$$\forall k.\ (RugbyPlayer(k) \land FootballPlayer(k) \to \neg 49er(k))$$

We could alternatively use this equivalence:

$$\neg(p \land q \land r) \equiv \neg p \lor \neg q \lor \neg r$$

This gives the following:

$$\neg\exists k.\ (RugbyPlayer(k) \land FootballPlayer(k) \land 49er(k))$$

$$\forall k.\ \neg(RugbyPlayer(k) \land FootballPlayer(k) \land 49er(k))$$

$$\forall k.\ (\neg RugbyPlayer(k) \lor \neg FootballPlayer(k) \lor \neg 49er(k))$$

The problem with this new statement is that (at least in my opinion) it's a lot harder to interpret what it means than the initial statement we came up with. It's not wrong, though.

ii. $\forall t.\ (Edible(t) \land Nutritious(t) \to Cultivated(t))$

Here's one option:

$$\neg\forall t.\ (Edible(t) \land Nutritious(t) \to Cultivated(t))$$

$$\exists t.\ \neg(Edible(t) \land Nutritious(t) \to Cultivated(t))$$

$$\exists t.\ (Edible(t) \land Nutritious(t) \land \neg Cultivated(t))$$

This says “there's something that's edible and nutritious but not cultivated.”

iii. $\forall p. \big(Person(p) \to \big(\exists q. \big(Person(q) \land TallerThan(p, q)\big)\big) \lor \big(\exists q. \big(Person(q) \land TallerThan(q, p)\big)\big)\big)$

Here's one option:

$$\neg\forall p. (Person(p) \to (\exists q. (Person(q) \land TallerThan(p, q))) \lor (\exists q. (Person(q) \land TallerThan(q, p))))$$

$$\exists p. \neg(Person(p) \to (\exists q. (Person(q) \land TallerThan(p, q))) \lor (\exists q. (Person(q) \land TallerThan(q, p))))$$

$$\exists p. (Person(p) \land \neg((\exists q. (Person(q) \land TallerThan(p, q))) \lor (\exists q. (Person(q) \land TallerThan(q, p)))))$$

$$\exists p. (Person(p) \land \neg(\exists q. (Person(q) \land TallerThan(p, q))) \land \neg(\exists q. (Person(q) \land TallerThan(q, p))))$$

$$\exists p. (Person(p) \land (\forall q. \neg(Person(q) \land TallerThan(p, q))) \land \neg(\exists q. (Person(q) \land TallerThan(q, p))))$$

$$\exists p. (Person(p) \land (\forall q. (Person(q) \to \neg TallerThan(p, q))) \land \neg(\exists q. (Person(q) \land TallerThan(q, p))))$$

$$\exists p. (Person(p) \land (\forall q. (Person(q) \to \neg TallerThan(p, q))) \land (\forall q. \neg(Person(q) \land TallerThan(q, p))))$$

$$\exists p. (Person(p) \land (\forall q. (Person(q) \to \neg TallerThan(p, q))) \land (\forall q. (Person(q) \to \neg TallerThan(q, p))))$$

This says “there is someone where no one is taller than them and no one is shorter than them.”

iv. $\exists r. (Silly(r) \leftrightarrow \neg Serious(r))$

Here's one option:

$$\neg\exists r. (Silly(r) \leftrightarrow \neg Serious(r))$$

$$\forall r. \neg(Silly(r) \leftrightarrow \neg Serious(r))$$

$$\forall r. (Silly(r) \leftrightarrow \neg\neg Serious(r))$$

$$\forall r. (Silly(r) \leftrightarrow Serious(r))$$

Why we asked this question: It's really important to be able to negate first-order logic statements. You'll use this when reasoning about proofs by contradiction and contrapositive and when trying to disprove statements. We asked a similar question like this on the problem set and figured it would be a good idea to give you some more practice.

Problem Seven: The Epimenides Paradox

If Epimenides is lying, then we cannot conclude that all Cretans are always truthful. The issue here is that the negation of "all Cretans always lie" is "some Cretan sometimes tells the truth." If Epimenides lies, then we just know that some Cretan sometimes tells the truth. It just doesn't have to be Epimenides at this point in time. Therefore, Epimenides is lying.

Why we asked this question: Now that we're getting more practice working with first-order statements and first-order negations, we're hoping that you're getting more comfortable reasoning about English statements and their negations. This question was designed to see if you could reason about statements in natural language by thinking about the underlying logical structure.

Problem Eight: Translating into Logic

i. Given the predicates *Orange*(*x*), which states that *x* is orange, and *Cat*(*x*), which states that *x* is a cat, write a formula in first-order logic that says "every cat is orange."

This nicely matches one of our Aristotelian forms: $\forall c. (Cat(c) \to Orange(c))$.

ii. Given the predicates *Orange*(*x*), which states that *x* is orange, and *Cat*(*x*), which states that *x* is a cat, write a formula in first-order logic that says "some cat is orange."

Another Aristotelian form: $\exists c. (Cat(c) \land Orange(c))$.

iii. Given the predicates *Orange*(*x*), which states that *x* is orange, and *Cat*(*x*), which states that *x* is a cat, write a formula in first-order logic that says "there are no orange cats."

Yet another Aristotelian form! $\forall c. (Cat(c) \to \neg Orange(c))$.

iv. Given the predicates *Orange*(*x*), which states that *x* is orange, and *Cat*(*x*), which states that *x* is a cat, write a formula in first-order logic that says "some cat is not orange."

The last of Aristotelian forms: $\exists c. (Cat(c) \land \neg Orange(c))$.

v. Given the predicates *Person*(x), which states that x is a person; *Orange*(x), which states that x is orange; *Cat*(x), which states that x is a cat; and *Likes*(x, y), which states that x likes y, write a formula in first-order logic that says "everyone likes at least one orange cat."

If we go step by step through a translation and use the Aristotelian forms as a guide, we get this:

$$\begin{array}{l} \forall p.\ (Person(p) \to \\ \quad \exists c.\ (Cat(c) \land Orange(c) \land Likes(p, c)) \\ ) \end{array}$$

This says "for any person p, there's an orange cat c that they like."

vi. Given the predicates *Person*(x), which states that x is a person; *Cat*(x), which states that x is a cat; and *Likes*(x, y), which states that x likes y, write a formula in first-order logic that says "everyone likes ***exactly one*** cat."

Using a combination of the Aristotelian forms and what we saw about uniqueness in lecture, we can come up with something like this:

$$\begin{array}{l} \forall p.\ (Person(p) \to \\ \quad \exists c.\ (Cat(c) \land Likes(p, c) \land \\ \quad\quad \forall d.\ (Cat(d) \land d \neq c \to \neg Likes(p, d)) \\ \quad) \\ ) \end{array}$$

This says "for any person p, there's a cat c that they like, and that person p doesn't like any other cats."

vii. Given the predicate *Person*(x), which states that x is a person, and *Muggle*(x), which states that x is a muggle, write a statement in first-order logic that says "some (but not all) people are muggles."

One possibility is given here, which says both that someone is a muggle and that someone is not a muggle.

$$\exists p.\ (Person(p) \land Muggle(p)) \quad \land \quad \exists p.\ (Person(p) \land \neg Muggle(p))$$

Another equivalent option is to say that someone is a muggle and that it's not the case that everyone is a muggle. Here's one way to do that:

$$\exists p.\ (Person(p) \land Muggle(p)) \quad \land \quad \neg\forall p.\ (Person(p) \to Muggle(p))$$

viii. Given the predicate *Person*(x), which states that x is a person, and *Ruler*(x), which states that x is a ruler, write a statement in first-order logic that says "at most one person is a ruler."

One way to express this idea is to say that either no one is a ruler, or there is just one person who's a ruler and everyone else isn't a ruler. This is shown here:

$$\begin{aligned}&\neg\exists p.\ (Person(p) \land Ruler(p)) \lor \\ &\exists p.\ (Person(p) \land Ruler(p) \land \\ &\quad \forall q.\ (Person(q) \land p \neq q \to \neg Ruler(q)) \\ &)\end{aligned}$$

Another option which works, but is a lot more subtle, is to say that there's some person where everyone who is a ruler is that one person. That way, if the person isn't a ruler, then no one is a ruler, and if that person is a ruler, then no one else is. This is shown here:

$$\begin{aligned}&\exists p.\ (Person(p) \land \\ &\quad \forall q.\ (Person(q) \land Ruler(q) \to p = q) \\ &) \lor \forall p.\ \bot\end{aligned}$$

This second one is a lot trickier to come up with and it's pretty subtle to see why it's correct, so don't worry if you didn't think of it. We thought we'd include it just for the sake of completeness. [Note: this version is subtle, and was originally incorrect in these solutions! Existential statements are always false in empty worlds, so we added the part in red to ensure the proposition would be true for empty worlds as well].

ix. Given the predicate *Instant*(i), which states that i is an instant in time, and *Precedes*(x, y), which states that x precedes y, write a sentence in first-order logic that says "time has a beginning, but has no end."

If there's a beginning of time, there's a point in time before all other time instants. Since there's no end to time, for every time point, there's another time point that comes after it. That gives us the following:

$$\begin{aligned}&\exists b.\ (Instant(b) \land \\ &\quad \forall x.\ (Instant(x) \land b \neq x \to Precedes(b, x)) \\ &) \land \\ &\forall b.\ (Instant(b) \to \\ &\quad \exists x.\ (Instant(x) \land Precedes(b, x)) \\ &)\end{aligned}$$

Why we asked this question: First-order logic is a rather expressive language for reasoning about mathematical statements, but the structure of first-order statements often looks quite different from their English equivalents. We wanted to provide you more practice with first-order translations so that you'd feel more comfortable working with first-order logic in the future.