Problems for Week Four

Problem One: Concept Checks

You know the drill. Here's a review from the topics from last week.

i. Give two examples of binary relations over the set $\mathbb{N}$.

ii. What three properties must a binary relation have to have in order to be an equivalence relation? Give the first-order definitions of each of those properties. For each definition of a property, explain how you would write a proof that a binary relation R has that property.

iii. If R is an equivalence relation over a set A and a is an element of A, what does the notation $[a]_R$ mean? Intuitively, what does it represent?

iv. What does the notation $f : A \to B$ mean?

v. Let $f : A \to B$ be a function. Express, in first-order logic, what property f has to satisfy to be an injection. Then, based on the structure of that formula, explain how you would write a proof that f is injective.

vi. Negate your statement from part (v) and simplify it as much as possible. Then, based on the structure of your formula, explain how you would write a proof that f is not injective.

vii. Let $f : A \to B$ be a function. Express, in first-order logic, what property f has to satisfy to be a surjection. Then, based on the structure of that formula, explain how you would write a proof that f is surjective.

viii. Negate your statement from part (vii) and simplify it as much as possible. Then, based on the structure of your formula, explain how you would write a proof that f is not surjective.

ix. Let $f : A \to B$ be a function. What properties must f have to be a bijection? How would you write a proof that f is bijective?

x. What would you need to prove to show that f is not a bijection?

xi. Let $f : A \to B$ and $g: B \to C$ be functions. What does the notation $f \circ g$ mean? How would you evaluate $(f \circ g)(x)$?

Problem Two: Equivalence Relations

This question explores various properties of equivalence relations.

i. In lecture, we proved that the binary relation ~ over $\mathbb{Z}$ defined as follows is an equivalence relation:

$$a \sim b \quad \text{if } a+b \text{ is even.}$$

Consider this new relation # defined over $\mathbb{Z}$:

$$a \# b \quad \text{if } a+b \text{ is odd.}$$

Is # an equivalence relation? If so, prove it. If not, disprove it.

ii. How many equivalence classes are there for the ~ relation defined above? What are they?

Problem Three: Inverse Relations

Let R be a binary relation over a set A. We can define a new relation over A called the ***inverse relation of R***, denoted R^{-1}, as follows:

$$xR^{-1}y \quad \text{if} \quad yRx$$

This question explores properties of inverse relations.

i. What is the inverse of the < relation over $\mathbb{Z}$? Briefly justify your answer.

ii. What is the inverse of the = relation over $\mathbb{Z}$? Briefly justify your answer.

iii. Prove or disprove: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

Problem Four: Monotone Functions

A function $f : \mathbb{R} \to \mathbb{R}$ is called ***monotone increasing*** if the following is true:

$$\forall x \in \mathbb{R}.\ \forall y \in \mathbb{R}.\ (x < y \to f(x) < f(y)).$$

This problem explores properties of monotone increasing functions.

i. Prove or disprove: every monotone increasing function is injective.

ii. Prove or disprove: every injective function from $\mathbb{R}$ to $\mathbb{R}$ is monotone increasing.

Problem Five: Involutions

A function $f : A \to A$ is called an ***involution*** if $f(f(x)) = x$ for all $x \in A$.

i. Find three different examples of involutions from $\mathbb{Z}$ to $\mathbb{Z}$. Briefly justify your answers.

ii. Prove that if f is an involution, then f is a bijection.

Problem Six: Functions and Relations – Together!

Let $f : A \to B$ be an arbitrary function. Define a new binary relation ~ over A as follows:

$$x \sim y \quad \text{if} \quad f(x) = f(y)$$

Prove that ~ is an equivalence relation over A.