Problems for Week Five

Problem One: Cardinality Concept Checks

Here's some quick concept checks to make sure we're all on the same page.

i. If A and B are sets, what is the formal definition of the statement $|A| = |B|$?

ii. If A and B are sets, what is the formal definition of the statement $|A| \neq |B|$?

Problem Two: Finding Functions

There are a lot of functions out there with interesting properties. This question asks you to think about different possible functions and what they might look like.

i. Find a function $f : \mathbb{N} \to \mathbb{N}$ that is both injective and surjective. Prove it meets those criteria.

ii. Find a function $g : \mathbb{N} \to \mathbb{N}$ that is injective but not surjective. Prove it meets those criteria.

iii. Find a function $h: \mathbb{N} \to \mathbb{N}$ that is not injective but is surjective. Prove it meets those criteria.

iv. Find a function $k : \mathbb{N} \to \mathbb{N}$ that's neither injective nor surjective. Prove it meets those criteria.

v. Based on your answers to these problems, explain why if you have a function $m : A \to B$ and you know that $|A| = |B|$, you cannot necessarily say anything about whether m is injective, surjective, or bijective.

Problem Three: $\aleph_0 \pm 1$

We use the symbol $\aleph_0$ to denote the cardinality of the set $\mathbb{N}$. What does $\aleph_0 - 1$ look like? This would be the cardinality of a set formed by starting with the natural numbers and deleting one of the elements.

Let's define S to be the set $\mathbb{N} - \{0\}$.

i. Briefly describe the set S in plain English.

ii. Find a way of pairing elements of S with elements of $\mathbb{N}$ so that no elements are uncovered.

iii. Based on your answer to part (vi) of this problem, define a bijection $f : S \to \mathbb{N}$.

iv. Prove that the function you found in part (iii) of this problem is a bijection. Since the cardinality of S is $\aleph_0 - 1$, this proves that $\aleph_0 - 1 = \aleph_0$.

The value $\aleph_0 + 1$ is the cardinality of a set formed by starting with $\mathbb{N}$ and adding in another value. So let's have T be the set $\mathbb{N} \cup \{\bigstar\}$, where $\bigstar$ is some arbitrarily-chosen object that isn't a natural number.

v. Briefly describe the set T in plain English.

vi. Find a way of pairing elements of T with elements of $\mathbb{N}$ so that no elements are uncovered.

vii. Based on your answer to part (ii) of this problem, define a bijection $g : T \to \mathbb{N}$.

viii. Prove that the function you found in part (iii) of this problem is a bijection. Since the cardinality of T is $\aleph_0 + 1$, this proves that $\aleph_0 + 1 = \aleph_0$.

Problem Four: Graph Theory Concept Checks

Here's a quick review of our concepts from graph theory.

i. What is an undirected graph? What is a directed graph?

ii. What does it mean for two nodes to be adjacent in a graph?

iii. What is a path in a graph? What is a simple path in a graph?

iv. What is a cycle in a graph?

v. What does it mean for two nodes to be connected in a graph?

vi. Is it possible for two nodes in a graph to be adjacent but not connected?

vii. Is it possible for two nodes in a graph to be connected but not adjacent?

viii. What does it mean for a graph G to be connected?

ix. What is a connected component in a graph?

x. How many connected components does each node in a graph belong to?

xi. What is a planar graph?

xii. What is a k-vertex-coloring of a graph?

xiii. What does the notation $\chi(G)$ mean?

xiv. What is meant by the degree of a node in a graph?

Problem Five: Graph Coloring

This problem explores some properties of graph coloring that may help you get a better intuitive feel for how colorings and node degrees relate to one another.

i. Give an example of a 2-colorable graph where some node has degree seven. Briefly justify why your graph meets these criteria; no proof is necessary.

ii. Generalize your answer from part (i) by describing how, for any $n \geq 0$, you can build a 2-colorable graph where some node has degree at least n. This shows that there is no direct connection between the maximum degree of a node in a graph and the chromatic number of that graph.

iii. Give an example of a 2-colorable graph where every node has degree three. Briefly justify why your graph meets these criteria; no proof is necessary.

iv. Generalize your answer from part (iii) by describing how, for any $n \geq 0$, you can build a 2-colorable graph where every node has degree at least n. This shows that there is no direct connection between the minimum degree of a node in the graph and the chromatic number of that graph.

v. Give an example of a graph where every node has degree two but which is not 2-colorable. Briefly justify why your graph meets these criteria; no proof is necessary.

vi. Generalize your answer from part (v) by describing how, for any $n \geq 0$, you can build a connected graph with *at least* n nodes where every node has degree two but which is not 2-colorable. This shows that each part of a graph can look 2-colorable even though the graph as a whole is not.

Problem Six: The Pigeonhole Principle

Suppose you pick 11 numbers from the list 1, 2, 3, 4, 5, …, 20. Prove that you must have chosen a pair of numbers whose difference is 10.