Solutions for Week Five

Problem One: Cardinality Concept Checks

i. If A and B are sets, what is the formal definition of the statement $|A| = |B|$?

There is a bijection $f : A \to B$.

ii. If A and B are sets, what is the formal definition of the statement $|A| \neq |B|$?

There no bijection $f : A \to B$. Equivalently, any function $f : A \to B$ is not a bijection.

Problem Two: Finding Functions

i. Find a function $f : \mathbb{N} \to \mathbb{N}$ that is both injective and surjective. Prove it meets those criteria.

One option is the identity function $f(n) = n$. This is injective: consider any natural numbers n_1 and n_2 where $f(n_1) = f(n_2)$. Then by definition of f we see that $n_1 = n_2$, so f is injective. This function is also surjective: given any $n \in \mathbb{N}$, we see that there is some $m \in \mathbb{N}$ (namely, n) such that $f(m) = n$.

ii. Find a function $g : \mathbb{N} \to \mathbb{N}$ that is injective but not surjective. Prove it meets those criteria.

One option is $g(n) = 2n$. This function is injective: consider any natural numbers n_1 and n_2 where $g(n_1) = g(n_2)$. Then by definition of g we see that $2n_1 = 2n_2$, which in turn means that $n_1 = n_2$, so g is injective. However, g is not surjective. To see this, note that there is no $m \in \mathbb{N}$ where $f(m) = 1$, since $f(m) = 2m$ is always even and 1 is odd.

iii. Find a function h: $\mathbb{N} \to \mathbb{N}$ that is not injective but is surjective. Prove it meets those criteria.

One option is $h(n) = \lfloor n/2 \rfloor$. This function is not injective: notice that $h(0) = \lfloor 0/2 \rfloor = 0$ and that $h(1) = \lfloor 1/2 \rfloor = 0$. However, it is surjective. Given any $n \in \mathbb{N}$, notice that $h(2n) = \lfloor 2n/2 \rfloor = \lfloor n \rfloor = n$.

iv. Find a function $k : \mathbb{N} \to \mathbb{N}$ that's neither injective nor surjective. Prove it meets those criteria.

One option is $k(n) = 0$. This function is not injective because $k(0) = k(1) = 0$. This function is also not surjective because there is no $m \in \mathbb{N}$ such that $f(m) = 1$.

v. Based on your answers to these problems, explain why if you have a function $m : A \to B$ and you know that $|A| = |B|$, you cannot necessarily say anything about whether m is injective, surjective, or bijective.

We know that $|\mathbb{N}| = |\mathbb{N}|$, but there are functions from $\mathbb{N}$ to $\mathbb{N}$ that are bijective, injective but not surjective, surjective but not injective, and neither injective nor surjective. Therefore, just choosing a function $f : \mathbb{N} \to \mathbb{N}$ tells us nothing about $\mathbb{N}$ even though we know $|\mathbb{N}| = |\mathbb{N}|$.

Problem Three: $\aleph_0 \pm 1$

Now, let's define S to be the set $\mathbb{N} - \{0\}$.

i. Briefly describe the set S in plain English.

This is the set of all natural numbers except 0.

ii. Find a way of pairing elements of S with elements of $\mathbb{N}$ so that no elements are uncovered.

Here's one option:

$$1 \leftrightarrow 0$$
$$2 \leftrightarrow 1$$
$$3 \leftrightarrow 2$$
$$4 \leftrightarrow 3$$
$$5 \leftrightarrow 4$$
$$\dots$$

iii. Based on your answer to part (vi) of this problem, define a bijection $f : S \to \mathbb{N}$.

One option is $f(n) = n - 1$.

iv. Prove that the function you found in part (iii) of this problem is a bijection. Since the cardinality of S is $\aleph_0 - 1$, this proves that $\aleph_0 - 1 = \aleph_0$.

Proof: We will prove that $|S| = |\mathbb{N}|$ by giving a bijection $f : S \to \mathbb{N}$. Let $g(n) = n - 1$. We will prove that g is a bijection by proving that f is injective and surjective.

To see that g is injective, consider any $n_1, n_2 \in S$ where $f(n_1) = f(n_2)$. We'll show that $n_1 = n_2$. To see this, note that since $f(n_1) = f(n_2)$, we know that $n_1 - 1 = n_2 - 1$. This means that $n_1 = n_2$, as required.

To see that f is surjective, consider any $n \in \mathbb{N}$. We need to show that there is some $m \in S$ such that $f(m) = n$. If we choose $m = n + 1$, then $f(m) = f(n + 1) = (n + 1) - 1 = n$, as required. ■

Let's have T be the set $\mathbb{N} \cup \{★\}$, where ★ is some arbitrarily-chosen object that isn't a natural number.

v. Briefly describe the set T in plain English.

T is the set of all natural numbers, plus ★.

vi. Find a way of pairing elements of T with elements of $\mathbb{N}$ so that no elements are uncovered.

Here's one way to do this:

$$\begin{gathered} ★ \leftrightarrow 0 \\ 0 \leftrightarrow 1 \\ 1 \leftrightarrow 2 \\ 2 \leftrightarrow 3 \\ \dots \end{gathered}$$

vii. Based on your answer to part (ii) of this problem, define a bijection $g : S \to \mathbb{N}$.

One possible function for the above pairing is shown here:

$$g(n) = \begin{cases} 0 & \text{if } n = ★ \\ n + 1 & \text{otherwise} \end{cases}$$

viii. Prove that the function you found in part (iii) of this problem is a bijection. Since the cardinality of T is $\aleph_0 + 1$, this proves that $\aleph_0 + 1 = \aleph_0$.

Proof: Consider the function $g : T \to \mathbb{N}$ defined as follows:

$$g(n) = \begin{cases} 0 & \text{if } n = ★ \\ n + 1 & \text{otherwise} \end{cases}$$

We will prove that g is a bijection by showing it is injective and surjective, from which we can conclude that $|T| = |\mathbb{N}|$.

To see that g is injective, consider any $n_1, n_2 \in T$ where $n_1 \neq n_2$. We will prove $g(n_1) \neq g(n_2)$. Since $n_1 \neq n_2$, at most one of n_1 and n_2 can be ★. We consider two cases:

Case 1: Neither $n_1 = ★$ nor $n_2 = ★$. Then $g(n_1) = n_1 + 1$ and $g(n_2) = n_2 + 1$, and since $n_1 \neq n_2$, we see $n_1 + 1 \neq n_2 + 1$. Thus $g(n_1) \neq g(n_2)$, as required.

Case 2: Exactly one of n_1 and n_2 is ★. Without loss of generality, assume that $n_1 = ★$ and that $n_2 \neq ★$. Then $g(n_1) = 0$ and $g(n_2) = n_2 + 1$. Since $n_2 \in \mathbb{N}$ and $f(n_2) = n_2 + 1$, we can see that $g(n_2) = n_2 + 1 \geq 1 > 0 = g(n_1)$, and so $g(n_1) \neq g(n_2)$, as required.

In both cases we see $g(n_1) \neq g(n_2)$, so g is injective, as required.

To see that g is surjective, consider any $n \in \mathbb{N}$. We will prove there is at least one $m \in T$ such that $g(m) = n$. If $n = 0$, then we can take $m = ★$, since $g(m) = g(★) = 0 = n$. Otherwise, we know that $n > 0$. This means $n \geq 1$, and so $n - 1$ is also a natural number. Taking $m = n - 1$ then guarantees that $g(m) = g(n - 1) = (n - 1) + 1 = n$, as required. Thus g is surjective. ■

Problem Four: Graph Theory Concept Checks

Here's a quick review of our concepts from graph theory.

i. What is an undirected graph? What is a directed graph?

An undirected graph is a set of nodes and a set of edges, where each edge is an unordered pair of elements drawn from the set of nodes. Formally, an undirected graph is a pair $G = (V, E)$ where E is a set of unordered pairs whose elements are drawn from V.

A directed graph is a set of nodes and a set of edges, where each edge is a directed edge from one node to another. Formally, a directed graph is a pair $G = (V, E)$ where E is a set of ordered pairs whose elements are drawn from V.

ii. What does it mean for two nodes to be adjacent in a graph?

Two nodes are adjacent if they're linked by an edge. Formally, if $G = (V, E)$ is a graph, then nodes $u, v \in V$ are adjacent if $\{u, v\} \in E$.

iii. What is a path in a graph? What is a simple path in a graph?

A path in a graph is a series of nodes where any two nodes in the path are adjacent. A simple path is one that doesn't repeat any nodes or edges.

iv. What is a cycle in a graph? What is a simple cycle in a graph?

A cycle in a graph is a path that starts and ends at the same node.

v. What does it mean for two nodes to be connected in a graph?

Two nodes are connected if there's a path from the first node to the second.

vi. Is it possible for two nodes in a graph to be adjacent but not connected?

No, that's not possible, because the single edge linking those nodes forms a path between them, making them connected.

vii. Is it possible for two nodes in a graph to be connected but not adjacent?

Yes. Consider a graph consisting of three nodes linked in a line. The first and last node on the line are connected (there's a path between them) but not adjacent.

viii. What does it mean for a graph G to be connected?

A graph G is connected if for any two nodes in G, those nodes are connected.

ix. What is a connected component in a graph?

A connected component in a graph is a set of nodes in the graph where (1) the set is nonempty, (2) any two nodes in the set are connected, and (3) any node in the set is not connected to any node outside the set.

x. How many connected components does each node in a graph belong to?

Each node in a graph belongs to exactly one connected component in that graph.

xi. What is a planar graph?

A planar graph is a graph that can be drawn in two dimensions with no edges crossing.

xii. What is a k-vertex-coloring of a graph?

A k-vertex-coloring of a graph is a way of coloring each node in the graph one of k different colors such that no two adjacent nodes are the same color (or, equivalently, so that each edge's endpoints are different colors.)

xiii. What does the notation $\chi(G)$ mean?

This is the chromatic number of G, the minimum value of k such that a k-vertex-coloring exists for G.

xiv. What is meant by the degree of a node in a graph?

The degree of a node in a graph is the number of edges touching it – equivalently, it's the number of nodes it's adjacent to.

Problem Five: Graph Coloring

i. Give an example of a 2-colorable graph where some node has degree seven. Briefly justify why your graph meets these criteria; no proof is necessary.

Consider a graph with nodes {1,2,3,4,5,6,7,8} where node 1 is adjacent to every other node (and there are no other edges). This graph is 2-colorable because we can color node 1 red and every other node blue. Additionally, the node 1 has degree seven.

ii. Generalize your answer from part (i) by describing how, for any $n \geq 0$, you can build a 2-colorable graph where some node has degree at least n. This shows that there is no direct connection between the maximum degree of a node in a graph and the chromatic number of that graph.

One option is a ***star graph*** consisting of a central node connected to n peripheral nodes each are only adjacent to the central node. This is 2-colorable because we can color the central node blue and every other node red. The central node in this graph has degree n.

iii. Give an example of a 2-colorable graph where every node has degree three. Briefly justify why your graph meets these criteria; no proof is necessary.

One option is the utility graph from lecture. It's 2-colorable (color all of the nodes on the top red and all the nodes on the bottom blue), and every node has degree three.

iv. Generalize your answer from part (iii) by describing how, for any $n \geq 0$, you can build a 2-colorable graph where every node has degree at least n. This shows that there is no direct connection between the minimum degree of a node in the graph and the chromatic number of that graph.

One option is ***complete bipartite graphs***. Given $n \geq 0$, build a graph of $2n$ nodes split into two groups of equal size. Link each node in each half with each node in the other half. Every node has degree n and the graph is 2-colorable: we can color one of the halves blue and the other half red.

v. Give an example of a graph where every node has degree two but which is not 2-colorable. Briefly justify why your graph meets these criteria; no proof is necessary.

Consider a cycle of three nodes. If we try to 2-color it, we'll have to assign the same color to two different nodes, which will be a problem because all nodes are connected to one another. However, it is 3-colorable because we can give each node its own color.

vi. Generalize your answer from part (v) by describing how, for any $n \geq 0$, you can build a connected graph with at least n nodes where every node has degree two but which is not 2-colorable. This shows that each part of a graph can look 2-colorable even though the graph as a whole is not.

Consider a cycle with $2n + 3$ nodes. This graph is not 2-colorable. To see this, fix any node in the cycle. The colors of the nodes going around both halves of the cycle must alternate from this color. However, when the halves meet in the middle, each half of the cycle will have passed through $n+1$ nodes, so the nodes at the meeting point must have the same color. If we use three colors, though, we can color the nodes in the cycle by assigning one of these two adjacent nodes the unused third color.

Problem Six: The Pigeonhole Principle

Suppose you pick 11 numbers from the list 1, 2, 3, 4, 5, …, 20. Prove that you must have chosen a pair of numbers whose difference is 10.

Proof: Consider the ten "buckets" {1, 11}, {2, 12}, ..., {10, 20}. There are ten buckets here. If we choose eleven numbers from the list 1, 2, 3, ..., 20, then if we distribute those numbers across the buckets then, each number belongs to exactly one bucket, the pigeonhole principle at least two of the numbers will land in the same bucket. By construction, the two numbers in each bucket differ by exactly 10, so our list must contain a pair whose difference is 10. ■