Problems for Week Six

This week is all about induction in its many forms. We hope these problems give you more practice working with induction and thinking inductively!

Getting Started: Induction and Fibonacci Numbers

The *Fibonacci numbers* are a series of numbers defined by a recurrence relation. The first two Fibonacci numbers are 0 and 1, and each number after that is defined as the sum of the two previous numbers. Formally speaking, we define the Fibonacci numbers as follows:

$$F_0 = 0 \qquad F_1 = 1 \qquad F_{n+2} = F_n + F_{n+1}$$

i. Using this definition, determine the values of F_2, F_3, F_4, F_5, F_6, and F_7.

The Fibonacci numbers have a lot of useful properties, many of which are most easily proven using mathematical induction. As an induction warm-up, we're going to have you prove a few of these properties. First, we're going to have you prove that for any natural number *n*, the following is true:

$$F_0 + F_1 + F_2 + F_3 + \ldots + F_n = F_{n+2} - 1.$$

We're going to have you work through this as a proof by induction. To do so, we're going to need some property $P(n)$ that we'll show is true for all $n \in \mathbb{N}$. We'll use this one here:

Let $P(n)$ be the statement "$F_0 + F_1 + F_2 + F_3 + \ldots + F_n = F_{n+2} - 1$."

ii. If you want to prove this property by induction, you will need to prove a base case. Write out what you need to prove in order to prove $P(0)$, then go prove it. *(Hint: How many terms will be in the summation when n = 0?)*

iii. In a proof by induction, you will assume that $P(k)$ is true for some $k \in \mathbb{N}$, then prove that $P(k+1)$ is true. Write out what it is that you'd be assuming if you assumed $P(k)$ is true, then write out what you need to prove in order to prove $P(k+1)$.

iv. Prove that if $P(k)$ is true, then $P(k+1)$ is true.

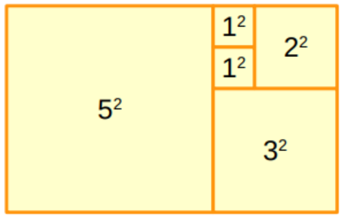
Congratulations! You've just worked through a full inductive proof. ☺

More Fun with Fibonacci Numbers

You just reasoned about the sums of Fibonacci numbers. It turns out that the sums of *squares* of Fibonacci numbers also have a bunch of nice properties. Specifically, for $n \geq 0$, we have

$$F_0^2 + F_1^2 + F_2^2 + \ldots + F_n^2 = F_n F_{n+1}.$$

Here's a graphical intuition for where this comes from:



i. Prove by induction that $F_0^2 + F_1^2 + F_2^2 + \ldots + F_n^2 = F_n F_{n+1}$ for all natural numbers n.

Here's another fun place where the Fibonacci numbers show up. Consider the following series of fractions:

$$1, \quad 1 + \frac{1}{1}, \quad 1 + \frac{1}{1 + \frac{1}{1}}, \quad 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}}, \quad 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}}}, \quad \ldots$$

These fractions are given by the recurrence relation

$$R_0 = 1 \qquad R_{n+1} = 1 + {}^{1}/_{R_n}.$$

It might not immediately be obvious why this is the case, so take a few minutes to go and check why this recurrence gives the above series.

ii. Give exact values for R_0, R_1, R_2, R_3, R_4, and R_5.

iii. You should see some sort of pattern emerge relating the numbers R_n from the series above to the Fibonacci numbers (hint: write the numbers as fractions). Fill in the blank below to indicate what that pattern is.

$$R_n = \underline{\hspace{6cm}}$$

iv. Using induction, prove that the pattern you came up with in part (iii) is correct.

The fractions here you've explored are called continued fractions and have all sorts of fun and exciting mathematical properties. Check out the Wikipedia article for more details!

Medicine Half-Lives

A doctor has prescribed a patient medicine that is absorbed into the bloodstream. The medicine has a *half-life* of one hour, meaning that each hour, half of the medicine in the patient's bloodstream will be removed by her body. For example, if the patient had 5mg of the medicine in her bloodstream at 6:00PM, then at 7:00PM she would have 2.5mg of the medicine in her bloodstream.

Suppose that the doctor gives the patient 1mg of the medicine at the start of every hour, all of which is immediately absorbed into her bloodstream. You are concerned because each time the patient receives a dose, some amount of the medicine from the previous doses will still be left in her bloodstream. Wouldn't this give the patient a dangerous amount of medicine?

Fortunately, now that you've taken a course in discrete math, you can determine exactly how much medicine will be in the patient's bloodstream, which will help you determine whether she will ever have a dangerous amount of the medicine in her blood.

Let c_n denote the amount of active medication in the patient's body n hours after the first dose has been administered. The first dose is 1mg, so c_0 = 1mg. One hour later, half of the medicine will have been cleared from her bloodstream, leaving 0.5mg, and the patient will receive 1mg more medicine, bringing the total up to 1.5mg. Thus c_1 = 1.5mg. An hour after that, half that medicine will have been cleared from her bloodstream, leaving 0.75mg of medicine in her bloodstream, and the patient will then receive another 1mg of medicine, bringing the total up to 1.75mg. Thus c_2 = 1.75mg.

i. Write a recurrence relation for c_n along these lines. That is, give a value for c_0, then express the value of c_{n+1} in terms of c_n.

ii. Using your recurrence relation from part (i), prove, by induction, that $c_n = (2 - 1/2^n)$mg for all $n \in \mathbb{N}$. This proves that the patient will never have more than 2mg of medicine in her bloodstream, even if she continues to take 1mg doses every hour.

Picking Coins

Consider the following game for two players. Begin with a pile of n coins for some $n \geq 0$. The first player then takes between one and ten coins out of the pile, then the second player takes between one and ten coins out of the pile. This process repeats until some player has no coins to take; at this point, that player loses the game.

Prove that if the pile begins with a multiple of eleven coins in it, the second player can always win if she plays correctly. To solve this problem, we recommend that you play this game against a friend until you can find the winning strategy. To prove this result, proceed by induction. You might want to take steps of an unusual size.

Factorials! Multiplied together!

If n is a natural number, then ***n factorial***, denoted ***n*!**, intuitively represents the product $1 \times 2 \times \ldots \times n$. Formally, we define $n!$ using a recurrence relation:

$$0! = 1 \qquad\qquad (n+1)! = (n+1) \cdot n!$$

i. What are 0!, 1!, 2!, 3!, 4!, and 5!?

We're going to ask you to prove the following result by induction:

$$\text{For any } m, n \in \mathbb{N}, \text{ we have } m!n! \leq (m+n)!. \qquad (\bigstar)$$

It might not be clear exactly how you would prove this result by induction, since there are two different variables involved here, m and n. The trick is to define a property $P(n)$ as follows:

Let $P(n)$ be the statement "for any $m \in \mathbb{N}$, we have $m!n! \leq (m+n)!$."

We can now try to prove $P(n)$ is true for all $n \in \mathbb{N}$ by induction on n.

ii. Explain why if we prove $P(n)$ is true for all $n \in \mathbb{N}$, we will prove that statement (★) is true.

iii. Prove, by induction, that $P(n)$ is true for all $n \in \mathbb{N}$. Go slowly through this proof – there are a lot of quantifiers here, so take the time to write out what you're assuming for $P(k)$ and what you need to prove in order to show that $P(k+1)$ is true.

iv. Give an intuitive explanation for why statement (★) is true without appealing to induction.

Elimination Tournaments

An *elimination tournament* is a contest between 2^n players that consists of a number of rounds. In each round, each player is paired with another player. Those players play a game and the loser of each game is eliminated from the tournament. When only one player remains, that player wins.

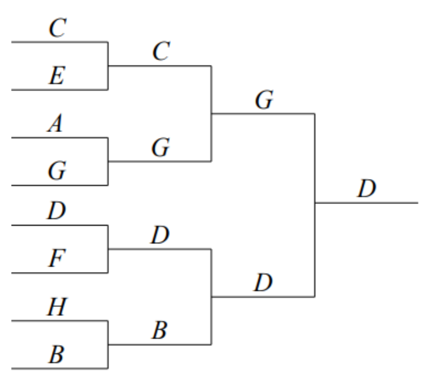
One of the advantages of elimination tournaments is that they can quickly determine a winner without everyone having to play everyone else.

i. Prove, by induction, that in an elimination tournament of 2^n players, exactly $2^n - 1$ total games are played. Then, give an intuitive justification for this result that doesn't use induction.

	A	***B***	***C***	***D***	***E***	***F***	***G***	***H***
A		W	W	W	W	W	L	W
B	L		W	L	L	L	W	L
C	L	L		W	W	W	L	W
D	L	W	L		L	W	W	L
E	L	W	L	W		W	W	W
F	L	W	L	L	L		W	L
G	W	L	W	L	L	L		L
H	L	W	L	W	L	W	W	

This savings comes at a cost, though. It turns out that if you have advance knowledge of which players would win or lose in a matchup against one another, you can sometimes rig tournaments so that a fairly weak player will end up winning. For example, suppose that you have eight players in a tournament, conveniently named *A*, *B*, *C*, *D*, *E*, *F*, *G*, and *H*. You can imagine that there is a “hypothetical outcome matrix” that shows, for each possible matchup, who would win in that matchup. Those matchups might not actually come up in an elimination tournament, of course, since not everyone plays everyone else.

For example, the above matrix says that player *A* would win in a matchup against players *B*, *C*, *D*, *E*, *F*, and *H*, but would lose to player *G*. Player *C* would lose to players *A*, *B*, and *G*, but would win against players *D*, *E*, *F*, and *H*. Given these hypothetical results, it seems like player *A* is the strongest player here – she would win against everyone except player *G*. Player *D*, on the other hand, is not a very strong player – if he were to play each other player, he'd lose more than half of his matches. However, as the elimination tournament bracket given here shows, if we knew the above results in advance, we could rig the tournament so that player *D* ends up coming out on top, even though in a head-to-head match against a random player he would likely lose.



Interestingly, though, there are limits to how weak of a player can come out on top. For example, it's not possible to rig a tournament so that players *F* or *G* would come out on top.

ii. Let *p* be a player in an elimination tournament of 2^n total players (where $n \geq 1$). Prove, by induction, that if *p* would win against fewer than *n* players in a head-to-head matchup, then *p* cannot possibly win the elimination tournament.

Prime Numbers

A natural number $n \geq 2$ is called a ***composite number*** if there are natural numbers p and q where $pq = n$ and where both p and q are not 1 and not n. For example, 6 is composite because we can write $6 = 2 \cdot 3$. However, 7 is not composite, because the only two natural numbers whose product is 7 are 1 and 7. A natural number $n \geq 2$ is called a ***prime number*** if it's not composite.

There's an important result in discrete mathematics that says that every natural number can be written as a product of zero or more prime numbers. To understand what's even meant by "a product of zero numbers" or "a product of one number," we'll say that a product of zero numbers is *by definition* equal to 1 (much in the same way that the sum of zero numbers is *by definition* equal to 0), and a product of just one number is *by definition* equal to that number. This means, for example, that we can write 1 as the product of no numbers and write any natural number n as the product of just itself and nothing else.

Using a proof by complete induction, prove that every natural number $n \geq 1$ can be written as a product of zero or more prime numbers.