

Solutions for Week Six

Getting Started: Induction and Fibonacci Numbers

The *Fibonacci numbers* are a series of numbers defined by a recurrence relation. The first two Fibonacci numbers are 0 and 1, and each number after that is defined as the sum of the two previous numbers. Formally speaking, we define the Fibonacci numbers as follows:

$$F_0 = 0 \qquad F_1 = 1 \qquad F_{n+2} = F_n + F_{n+1}$$

- i. Using this definition, determine the values of $F_2, F_3, F_4, F_5, F_6,$ and F_7 .

We know that $F_2 = F_0 + F_1$, so $F_2 = 0 + 1 = 1$.

We know that $F_3 = F_2 + F_1$, so $F_3 = 1 + 1 = 2$.

We know that $F_4 = F_3 + F_2$, so $F_4 = 2 + 1 = 3$.

We know that $F_5 = F_4 + F_3$, so $F_5 = 3 + 2 = 5$.

We know that $F_6 = F_5 + F_4$, so $F_6 = 5 + 3 = 8$.

We know that $F_7 = F_6 + F_5$, so $F_7 = 8 + 5 = 13$.

Let $P(n)$ be the statement “ $F_0 + F_1 + F_2 + F_3 + \dots + F_n = F_{n+2} - 1$.”

- ii. If you want to prove this property by induction, you will need to prove a base case. Write out what you need to prove in order to prove $P(0)$, then go prove it.

Proof: We need to prove $P(0)$, that $F_0 = F_{0+2} - 1$. To see this, note that $F_0 = 0$ by definition, and $F_2 = 1$ as shown above. Therefore, $F_0 = 0$ and $F_2 - 1 = 1 - 1 = 0$, so $F_0 = F_2 - 1$, as required. ■

- iii. In a proof by induction, you will assume that $P(k)$ is true for some $k \in \mathbb{N}$, then prove that $P(k+1)$ is true. Write out what it is that you'd be assuming if you assumed $P(k)$ is true, then write out what you need to prove in order to prove $P(k+1)$.

We will assume for some arbitrary $k \in \mathbb{N}$ that $F_0 + F_1 + F_2 + F_3 + \dots + F_k = F_{k+2} - 1$. We will need to prove that $F_0 + F_1 + F_2 + F_3 + \dots + F_k + F_{k+1} = F_{k+3} - 1$.

iv. Prove that if $P(k)$ is true, then $P(k+1)$ is true.

Proof: Assume that, for some arbitrary $k \in \mathbb{N}$, we have $F_0 + F_1 + F_2 + F_3 + \dots + F_k = F_{k+2} - 1$. We will need to prove that $F_0 + F_1 + F_2 + F_3 + \dots + F_k + F_{k+1} = F_{k+3} - 1$. To do so, notice that

$$F_0 + F_1 + F_2 + F_3 + \dots + F_k + F_{k+1} = (F_0 + F_1 + F_2 + F_3 + \dots + F_k) + F_{k+1}. \quad (1)$$

Our inductive hypothesis tells us that the parenthesized quantity in the right-hand side of equation (1) is equal to $F_{k+2} - 1$. Substituting this in, we see that

$$F_0 + F_1 + F_2 + F_3 + \dots + F_k + F_{k+1} = F_{k+2} - 1 + F_{k+1}. \quad (2)$$

Since $F_{k+1} + F_{k+2} = F_{k+3}$, we can clean up the right-hand side of equation (2) to see that

$$F_0 + F_1 + F_2 + F_3 + \dots + F_k + F_{k+1} = F_{k+3} - 1, \quad (3)$$

which is what we needed to show. ■

Why we asked this question: As you could probably tell, this question was designed as a warm-up problem for mathematical induction. We chose this particular problem because it involves a recurrence relation (which you'll use on the current problem set) and because the inductive step requires using the inductive definition of the Fibonacci numbers. We broke the problem down the way we did to walk you through the structure of an induction proof one step at a time, at each point asking you what you needed to assume and what you needed to prove so that you would have a better sense for how everything fits together.

More Fun with Fibonacci Numbers

i. Prove by induction that $F_0^2 + F_1^2 + F_2^2 + \dots + F_n^2 = F_n F_{n+1}$ for all natural numbers n .

Proof: Let $P(n)$ be the statement “ $F_0^2 + F_1^2 + F_2^2 + \dots + F_n^2 = F_n F_{n+1}$.” We will prove by induction on n that $P(n)$ holds for all $n \in \mathbb{N}$, from which the theorem follows.

As our base case, we will prove $P(0)$, that $F_0^2 = F_0 F_1$. To see this, note that $F_0 = 0$, so $F_0^2 = 0$ and $F_0 F_1 = 0$. Therefore, we see $F_0^2 = F_0 F_1$, as required.

For our inductive step, assume for some arbitrary $k \in \mathbb{N}$ that $P(k)$ is true, meaning that

$$F_0^2 + F_1^2 + F_2^2 + \dots + F_k^2 = F_k F_{k+1}. \quad (1)$$

We need to prove that $P(k+1)$ is true, meaning that we need to show

$$F_0^2 + F_1^2 + F_2^2 + \dots + F_k^2 + F_{k+1}^2 = F_{k+1} F_{k+2}.$$

To see this, notice that

$$F_0^2 + F_1^2 + F_2^2 + \dots + F_k^2 + F_{k+1}^2 = (F_0^2 + F_1^2 + F_2^2 + \dots + F_k^2) + F_{k+1}^2. \quad (2)$$

Equation (1), our inductive hypothesis, tells us that the parenthesized summation on the right-hand side of equation (2) is equal to $F_k F_{k+1}$, so we can substitute this in to yield

$$F_0^2 + F_1^2 + F_2^2 + \dots + F_k^2 + F_{k+1}^2 = F_k F_{k+1} + F_{k+1}^2. \quad (3)$$

Starting with equation (3) and performing some algebra yields the following:

$$\begin{aligned} F_0^2 + F_1^2 + F_2^2 + \dots + F_k^2 + F_{k+1}^2 &= F_k F_{k+1} + F_{k+1}^2 \\ &= F_{k+1}(F_k + F_{k+1}) \\ &= F_{k+1} F_{k+2} \end{aligned} \quad (4)$$

Thus $F_0^2 + F_1^2 + F_2^2 + \dots + F_k^2 + F_{k+1}^2 = F_{k+1} F_{k+2}$, as required. ■

Consider the recurrence relation

$$R_0 = 1 \quad R_{n+1} = 1 + \frac{1}{R_n}.$$

ii. Give exact values for R_0, R_1, R_2, R_3, R_4 , and R_5 .

By definition, $R_0 = 1$.

This means that $R_1 = 1 + \frac{1}{R_0} = 1 + \frac{1}{1} = 2$.

This means that $R_2 = 1 + \frac{1}{R_1} = 1 + \frac{1}{2} = \frac{3}{2}$.

This means that $R_3 = 1 + \frac{1}{R_2} = 1 + \frac{1}{(3/2)} = 1 + \frac{2}{3} = \frac{5}{3}$.

This means that $R_4 = 1 + \frac{1}{R_3} = 1 + \frac{1}{(5/3)} = 1 + \frac{3}{5} = \frac{8}{5}$.

This means that $R_5 = 1 + \frac{1}{R_4} = 1 + \frac{1}{(8/5)} = 1 + \frac{5}{8} = \frac{13}{8}$.

iii. You should see some sort of pattern emerge relating the numbers R_n from the series above to the Fibonacci numbers. Fill in the blank below to indicate what that pattern is.

The pattern seems to be $R_n = F_{n+2} / F_{n+1}$.

iv. Using induction, prove that the pattern you came up with in part (iii) is correct.

Proof: Let $P(n)$ be the statement “ $R_n = F_{n+2} / F_{n+1}$.” We will prove by induction that $P(n)$ holds for all $n \in \mathbb{N}$, from which the theorem follows.

As our base case, we prove $P(0)$, that $R_0 = F_2 / F_1$. To see this, note that by definition $R_0 = 1$ and that both F_2 and F_1 are equal to 1. Since $1 = 1 / 1$, we see that $P(0)$ holds, as required.

For our inductive step, assume for some arbitrary $k \in \mathbb{N}$ that $P(k)$ is true, meaning $R_k = F_{k+2} / F_{k+1}$. We will prove $P(k+1)$, that $R_{k+1} = F_{k+3} / F_{k+2}$. To see this, notice that, by definition, we have

$$R_{k+1} = 1 + 1 / R_k. \quad (1)$$

By our inductive hypothesis, we know that $R_k = F_{k+2} / F_{k+1}$. Substituting this into equation (1) yields

$$R_{k+1} = 1 + 1 / (F_{k+2} / F_{k+1}). \quad (2)$$

From here, we can use some algebra to conclude the following:

$$\begin{aligned} R_{k+1} &= 1 + 1 / (F_{k+2} / F_{k+1}) \\ &= 1 + F_{k+1} / F_{k+2} \\ &= F_{k+2} / F_{k+2} + F_{k+1} / F_{k+2} \\ &= (F_{k+2} + F_{k+1}) / F_{k+2} \\ &= F_{k+3} / F_{k+2}. \end{aligned}$$

Thus $R_{k+1} = F_{k+3} / F_{k+2}$, as required. ■

Why we asked this question: These problems were designed to give you more practice with induction while building on the theme from the previous problem. We included part (i), an old problem set question, so that you could get some experience writing an inductive proof without any scaffolding, hopefully using your proof from the first question as a guide. Part (ii) of this problem was designed to get you to engage with a different recurrence relation. We asked parts (iii) and (iv) of this problem to give you a feel for what induction looks like in practice. Rarely are you given a statement in isolation and told to prove that it has some property. Instead, you often discover some sort of interesting pattern, then need to write a proof explaining why that particular pattern is not actually a coincidence. Our hope was that you were able to use everything you've seen about Fibonacci numbers so far to make the key insight in part (iii), leaving part (iv) as mostly an exercise in algebraic manipulations.

Medicine Half-Lives

- i. Write a recurrence relation for c_n .

One possible recurrence relation is

$$c_0 = 1 \text{ mg}$$

$$c_{n+1} = c_n / 2 + 1 \text{ mg}$$

This works because at time 0, the patient has 1mg of the medicine in her bloodstream, and each hour after the first, half the previous amount of medicine is cleared and 1mg new medicine is added.

- ii. Using your recurrence relation from part (i), prove, by induction, that $c_n = (2 - 1/2^n)\text{mg}$ for all $n \in \mathbb{N}$. This proves that the patient will never have more than 2mg of medicine in her bloodstream, even if she continues to take 1mg doses every hour.

Theorem: For every $n \in \mathbb{N}$, we have $c_n = (2 - 1/2^n)\text{mg}$.

Proof: Let $P(n)$ be the statement “ $c_n = (2 - 1/2^n)\text{mg}$.” We will prove by induction that $P(n)$ holds for all $n \in \mathbb{N}$, from which the theorem follows.

As a base case, we prove $P(0)$, that $c_0 = (2 - 1/2^0)\text{mg}$. By definition, $c_0 = 1\text{mg}$. Simplifying the right-hand side of the equality, we see that $(2 - 1/2^0)\text{mg} = (2 - 1)\text{mg} = 1\text{mg}$. Therefore, these quantities are equal to one another, so $P(0)$ holds.

For the inductive step, assume that $P(k)$ for some arbitrary $k \in \mathbb{N}$, meaning that $c_k = (2 - 1/2^k)\text{mg}$. We will prove that $P(k+1)$ is also true, meaning that $c_{k+1} = (2 - 1/2^{k+1})\text{mg}$. By using our inductive hypothesis and the definition of c_{k+1} , we see that

$$\begin{aligned} c_{k+1} &= c_k / 2 + 1 \text{ mg} \\ &= ((2 - 1/2^k) / 2)\text{mg} + 1 \text{ mg} \\ &= (1 - 1/2^{k+1})\text{mg} + 1 \text{ mg} \\ &= (2 - 1/2^{k+1})\text{mg} \end{aligned}$$

Therefore, $c_{k+1} = (2 - 1/2^{k+1})\text{mg}$, as required. Thus $P(k+1)$ is true, completing the induction. ■

Why we asked this question: Up to this point, we have not required you to write out your own recurrence relations. Here, we asked you to do that, then to write a proof about the structure of that recurrence relation.

Picking Coins

Consider the following game for two players. Begin with a pile of n coins for some $n \geq 0$. The first player then takes between one and ten coins out of the pile, then the second player takes between one and ten coins out of the pile. This process repeats until some player has no coins to take; at this point, that player loses the game.

Prove that if the pile begins with a multiple of eleven coins in it, the second player can always win if she plays correctly.

There are many ways we can prove this. This first proof works by using induction on steps of size eleven:

Theorem: If this game is played with the pile containing $11n$ coins for some natural number n , the second player can always win the game.

Proof 1: Let $P(n)$ be the statement “if the game is played with the pile containing n coins, the second player can always win if she plays correctly.” We will prove by induction that $P(n)$ holds for all natural numbers n that are multiples of 11, from which the theorem follows.

As a base case, we need to prove $P(0)$, that if the game is played with a pile containing 0 coins, the second player always can win. This is true because there are no coins in the pile when the game starts, and so no matter what the second player does, she'll win because the first player loses.

For the inductive step, assume for some arbitrary $k \in \mathbb{N}$ that $P(k)$ is true and if the game is played with k coins, the second player can always win if she plays correctly. We will prove that $P(k + 11)$ holds: the second player can always win in a game with $k + 11$ coins if she plays correctly

Suppose the game starts with $k + 11$ coins. Consider the first player's move, which must remove some number c coins from the pile, where $1 \leq c \leq 10$. This leaves $k+11-c$ coins remaining. Now, suppose the second player then removes $11-c$ coins from the pile. This leaves a total of $k+11-c-(11-c) = k$ coins remaining in the pile, and it's now the first player's turn again. By the inductive hypothesis, this means that the second player can force a win in this situation, so the second player will eventually win the game. Consequently, starting with $k+11$ coins, the second player can win, so $P(k+11)$ holds, completing the induction. ■

Here's another way to prove this that works by having $P(n)$ explicitly talk about multiples of eleven, then using regular induction with steps of size one:

Theorem: If this game is played with the pile containing $11n$ coins for some natural number n , the second player can always win the game.

Proof: Let $P(n)$ be the statement “if the game is played with the pile containing $11n$ coins, the second player can always win if she plays correctly.” We will prove by induction that $P(n)$ holds for all natural numbers $n \in \mathbb{N}$, from which the theorem follows.

As a base case, we need to prove $P(0)$, that if the game is played with a pile containing $11 \cdot 0 = 0$ coins, the second player always can win. This is true because there are no coins in the pile when the game starts, and so no matter what the second player does, she'll win because the first player loses.

For the inductive step, assume for some arbitrary $k \in \mathbb{N}$ that $P(k)$ is true and if the game is played with $11k$ coins, the second player can always win if she plays correctly. We will prove $P(k+1)$ holds: the second player can always win in a game with $11(k+1) = 11k + 11$ coins if she plays correctly

Suppose the game starts with $11k + 11$ coins. Consider the first player's move, which must remove some number c coins from the pile, where $1 \leq c \leq 10$. This leaves $11k+11-c$ coins remaining. Now, suppose the second player then removes $11-c$ coins from the pile. This leaves a total of $11k+11-c-(11-c) = 11k$ coins remaining in the pile, and it's now the first player's turn again. By the inductive hypothesis, this means that the second player can force a win in this situation, so the second player will eventually win the game. Consequently, starting with $11k+11$ coins, the second player can win, so $P(k+1)$ holds, completing the induction. ■

Why we asked this question: Notice how this proof works by explicitly stating the second player's strategy, which is to make the total number of coins removed by her move and the first player's move come out to 11. This enables the inductive hypothesis to guarantee that the strategy can then force a win from the previous multiple of eleven. When proving that a certain player can win a game given some setup, a common technique is to see if that player can react to the first player's turn in a way that reduces the game to a case that is already known to be solved.

The technique employed here, in which we use inductive reasoning to prove that a player has a certain strategy, can be generalized to a technique called *backwards induction*, which reasons about a situation by considering the very last action and then working backwards to see how each player would try to make that action work in their favor.

Factorials! Multiplied together!

If n is a natural number, then n *factorial*, denoted $n!$, intuitively represents the product $1 \times 2 \times \dots \times n$. Formally, we define $n!$ using a recurrence relation:

$$0! = 1 \qquad (n+1)! = (n+1) \cdot n!$$

- i. What are $0!$, $1!$, $2!$, $3!$, $4!$, and $5!$?

$0!$, by definition, is equal to 1.

We therefore have that $1! = 1 \cdot 0! = 1 \cdot 1 = 1$.

We therefore have that $2! = 2 \cdot 1! = 2 \cdot 1 = 2$.

We therefore have that $3! = 3 \cdot 2! = 3 \cdot 2 = 6$.

We therefore have that $4! = 4 \cdot 3! = 4 \cdot 6 = 24$.

We therefore have that $5! = 5 \cdot 4! = 5 \cdot 24 = 120$.

For any $m, n \in \mathbb{N}$, we have $m!n! \leq (m+n)!$. (★)

Let $P(n)$ be the statement “for any $m \in \mathbb{N}$, we have $m!n! \leq (m+n)!$.”

- ii. Explain why if we prove $P(n)$ is true for all $n \in \mathbb{N}$, we will prove that statement (★) is true.

If $P(n)$ is true for all natural numbers n , then it means that for any choice of n and for any choice of m , we have $m!n! \leq (m+n)!$. This is precisely what's articulated by statement (★).

iii. Prove, by induction, that $P(n)$ is true for all $n \in \mathbb{N}$. Go slowly through this proof – there are a lot of quantifiers here, so take the time to write out what you're assuming for $P(k)$ and what you need to prove in order to show that $P(k+1)$ is true.

Theorem: For any $m, n \in \mathbb{N}$, the inequality $m!n! \leq (m+n)!$ is true.

Proof: Let $P(n)$ be the statement “for any $m \in \mathbb{N}$, the inequality $m!n! \leq (m+n)!$ is true.” We will prove, by induction, that $P(n)$ is true for all $n \in \mathbb{N}$, from which the theorem follows.

For our base case, we prove $P(0)$, that for any $m \in \mathbb{N}$, we have $m!0! \leq (m+0)!$. Notice that

$$m!0! = m! \cdot 1 = m! = (m+0)!$$

Therefore, $m!0! \leq (m+0)!$.

For our inductive step, assume for some arbitrary $k \in \mathbb{N}$ that $P(k)$ holds, so for any $m \in \mathbb{N}$, we know that $m!k! \leq (m+k)!$. We will prove $P(k+1)$: for any $m \in \mathbb{N}$, we have $m!(k+1)! \leq (m+k+1)!$ To do this, begin with any $m \in \mathbb{N}$. Expanding out the definition of factorial and using our inductive hypothesis shows the following:

$$\begin{aligned} m!(k+1)! &= m!k!(k+1) && \text{(by definition of } (k+1)!) \\ &\leq (m+k)!(k+1) && \text{(by the inductive hypothesis)} \\ &< (m+k)!(m+k+1) && \text{(since } k+1 < m+k+1) \\ &= (m+k+1)! && \text{(by definition of } (m+k+1)!) \end{aligned}$$

This tells us that $m!(k+1)! \leq (m+k+1)!$. Therefore, $P(k+1)$ holds, completing the induction. ■

iv. Give an intuitive explanation for why statement (★) is true without appealing to induction.

Intuitively, you can see why this result would be true by expanding out the quantity $m!n!$ and the quantity $(m+n)!$ and putting them side-by-side. Notice that $m!n! = 1 \cdot 2 \cdot \dots \cdot m \cdot 1 \cdot 2 \cdot \dots \cdot n$. This should be no greater than the quantity $1 \cdot 2 \cdot \dots \cdot m \cdot (m+1) \cdot (m+2) \cdot \dots \cdot (m+n) = (m+n)!$. The induction formalizes this reasoning.

Why we asked this question: Notice how our property $P(n)$ internally ranges over all possible choices of m . It is often possible to do induction on claims involving multiple variables by doing normal induction on just one variable and proving the claim is true for the remaining variables as normal. This style of induction (where we do induction on one variable and use a proof of the form “pick an arbitrary...” on another) is quite common when working with multiple variables. We hoped this example gives some guidance on how to structure those sorts of proofs!

Elimination Tournaments

i. Prove, by induction, that in an elimination tournament of 2^n players, exactly $2^n - 1$ total games are played. Then, give an intuitive justification for this result that doesn't use induction.

Proof: Let $P(n)$ be the statement “in an elimination tournament of 2^n players, exactly $2^n - 1$ total games are played.” We will prove, by induction, that $P(n)$ holds for all $n \in \mathbb{N}$, from which the theorem follows.

As a base case, we will prove $P(0)$, that in an elimination tournament of $2^0 = 1$ player, exactly $2^0 - 1 = 0$ total games are played. Since there is only a single player, that player wins immediately without any games being played.

For the inductive step, assume for some arbitrary $k \in \mathbb{N}$ that $P(k)$ is true, meaning that in an elimination tournament of 2^k players, exactly $2^k - 1$ games are played. We will prove $P(k+1)$, that in an elimination tournament of 2^{k+1} players, exactly $2^{k+1} - 1$ games are played.

Consider an elimination tournament with 2^{k+1} players in it. In the first round, each of the 2^{k+1} players is paired off with another player, so 2^k games are played. Half of them lose their games and are eliminated, and the remaining 2^k players will then play in the remaining tournament. That remaining tournament is an elimination tournament with 2^k players, so $2^k - 1$ games will then be played. The total number of games played is then $2^k + 2^k - 1 = 2(2^k) - 1 = 2^{k+1} - 1$, as required. ■

Intuitively, each game eliminates one player and a total of $2^n - 1$ players need to be eliminated.

ii. Let p be a player in an elimination tournament of 2^n total players (where $n \geq 1$). Prove, by induction, that if p would win against fewer than n players in a head-to-head matchup, then p cannot possibly win the elimination tournament.

Proof: Let $P(n)$ be the statement “if a player in a 2^n -player elimination tournament would win against fewer than n of the other players in a matchup, that player cannot possibly win the tournament.” We will prove by induction that $P(n)$ holds for all $n \geq 1$, from which the theorem follows.

As a base case, we prove $P(1)$, that if a player in a 2-player elimination tournament would win against fewer than 1 other player, that player cannot possibly win the elimination tournament. In this setup, the player is guaranteed to lose their only game because they would win against at most 0 other players, so they will be eliminated in their first match, ensuring that they cannot win the tournament.

For the inductive step, assume for some arbitrary $k \in \mathbb{N}$, $k \geq 1$, that $P(k)$ is true and no player in any 2^k -player elimination tournament who would win fewer than k games can win the tournament. We'll prove $P(k+1)$, that no player in any 2^{k+1} -player tournament who would win fewer than $k+1$ games can win the tournament.

Consider any player in a 2^{k+1} -player tournament who can win against fewer than $k+1$ players. Look at their first round. If they're paired against someone they can't beat, then they lose the tournament. Otherwise, they're paired against someone they can beat. This means that after the round, they've eliminated one of the players they can beat, so there are fewer than k players remaining that they can beat. There are also 2^k players left. Therefore, by the inductive hypothesis, the player has no way of winning at this point. In all cases, the player will lose the tournament, as required. ■

Why we asked this question: We chose these problems to show that induction can be applied to all sorts of discrete structures, not just the vanilla natural numbers. Plus, this particular problem is something that I worked on as part of my Ph.D. research!

Prime Numbers

Using a proof by complete induction, prove that every natural number $n \geq 1$ can be written as a product of zero or more prime numbers.

Theorem: Every natural number greater than one can be written as the product of prime numbers.

Proof: Let $P(n)$ be the statement “ n can be written as the product of zero or more prime numbers.” We will prove by complete induction that $P(n)$ holds for all $n \geq 1$.

As a base case, we need to show $P(1)$, that 1 can be written as the product of primes. We can express 1 as the product of zero prime numbers, which is by definition equal to 1.

For the inductive step, assume for some arbitrary $k \geq 1$ that $P(1), P(2), \dots, P(k)$ hold. This means that any natural number between 1 and k , inclusive, can be written as the product of prime numbers. We will prove $P(k+1)$, that $k+1$ can be written as the product of primes.

Consider the number $k+1$. If $k+1$ is prime, then it is the product of just itself and we're done. Otherwise, $k+1$ is composite, so it can be written as pq for some natural numbers p and q where neither p nor q is one and neither p nor q is equal to $k+1$. This means that p and q must be between 2 and k , inclusive, so we can express p and q as products of primes. We can then express $k+1$ as a product of primes by multiplying together all the prime numbers in the expressions of p and q as prime numbers. Therefore, $P(k+1)$ is true, completing the induction. ■

Why we asked this question: Notice how complete induction is employed to its fullest here. A normal induction couldn't correctly conclude that p and q are the products of primes, since they could be much smaller than k . However, since we're using complete induction, any natural number between 1 and k inclusive is assumed to be the product of primes, so the proof works out correctly.

Going forward, this proof is one of the two parts of the *fundamental theorem of arithmetic* (FTA), which says that any natural number greater than one can be written *uniquely* as the product of primes. It is a cornerstone of number theory and is often used as a first example when discussing complete induction. In order to show that the representation is unique, you need another lemma called *Euclid's lemma*, which says that if a prime number divides some product mn , then either that prime divides m or it divides n .