Mathematical Logic

Where We Are Now

- Our coverage of logic focused on these topics:
  - Propositional variables.
  - Propositional connectives.
  - Propositional equivalences.
  - Predicates, functions, and constant symbols.
  - Objects and propositions.
  - Quantifiers.
  - Evaluating first-order formulas relative to a world.
  - Translating into first-order logic.
  - Negating and simplifying first-order formulas.
- Your goal this week is to keep your proof skills sharp while mastering the ins and outs of first-order logic.

Things You Should Do Today

- Review the solutions for Problem Set One and make sure you ***completely*** and ***unambiguously*** understand the answers. Ask for help if this isn't the case!
- Read the “Guide to Negating Formulas” and “Guide to First-Order Logic Translations” on the course website to get more exposure and practice with those skills.
- Continue working through PS2.

Things You Should Do Tomorrow

- Look over your feedback on PS1 (and the PS2 checkpoint) and make sure you understand all the feedback you get ***completely*** and ***unambiguously***. Ask the course staff for help, either on Piazza or in office hours, if you don't.
- Continue working on PS2.
- Start reviewing your partner's answers and compiling a single, definitive set of answers that you're going to turn in.
- Stop by office hours to get feedback on your proofs and take that feedback seriously.
- (Also, complete the ***CS103A assignment*** after Wednesday’s lecture—it is due Friday at 2:30 pm!)

Topics You Wanted More Practice With

- First-order logic and translations [today's topic]
- Truth tables
- Proofs and disproofs [more next week]
  - For now, make sure you look at the PS1 solutions and feedback!!

Mechanics: ***Negating Statements***

$$
\begin{aligned}
&\forall p.\ (Person(p) \rightarrow \\
&\quad \exists q.\ (Person(q) \land p \neq q \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\neg \forall p.\ (Person(p) \rightarrow \\
&\quad \exists q.\ (Person(q) \land p \neq q \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\neg \forall p.\ (Person(p) \to \\
&\quad \exists q.\ (Person(q) \land p \neq q \land \\
&\quad\quad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

Useful Resource:

Go to cs103.stanford.edu and read the Guide to Negating Formulas.

$\neg\forall p.\ (Person(p) \to$

$\quad \exists q.\ (Person(q) \land p \neq q \land$

$\qquad Loves(p, q)$

$\quad)$

$)$

$$\boxed{\dfrac{\neg\forall x.\ A}{\exists x.\ \neg A}}$$

$$
\begin{aligned}
&\neg\forall p.\ (Person(p) \rightarrow \\
&\quad \exists q.\ (Person(q) \land p \neq q \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\boxed{\dfrac{\neg\forall x.\ A}{\exists x.\ \neg A}}
$$

$$
\begin{aligned}
&\neg \forall p.\ (Person(p) \rightarrow \\
&\quad \exists q.\ (Person(q) \wedge p \neq q \wedge \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$\frac{\neg \forall x.\ A}{\exists x.\ \neg A}$$

$$
\begin{array}{l}
\exists p.\ \neg(Person(p) \rightarrow \\
\quad \exists q.\ (Person(q) \land p \neq q \land \\
\quad\quad Loves(p, q) \\
\quad) \\
)
\end{array}
$$

$$
\boxed{\dfrac{\neg\forall x.\ A}{\exists x.\ \neg A}}
$$

$$
\begin{aligned}
&\exists p.\ \neg(Person(p) \to \\
&\quad \exists q.\ (Person(q) \land p \neq q \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\exists p.\ \neg(\mathit{Person}(p) \to \\
&\quad \exists q.\ (\mathit{Person}(q) \land p \neq q \land \\
&\qquad \mathit{Loves}(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\boxed{\dfrac{\neg(A \to B)}{A \land \neg B}}
$$

$$
\begin{aligned}
&\exists p.\ \neg(\mathit{Person}(p) \to \\
&\quad \exists q.\ (\mathit{Person}(q) \land p \neq q \land \\
&\qquad \mathit{Loves}(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\frac{\neg(A \to B)}{A \land \neg B}
$$

$$\exists p.\ \neg(\mathit{Person}(p) \to \exists q.\ (\mathit{Person}(q) \land p \neq q \land \mathit{Loves}(p, q)))$$

$$\frac{\neg(A \to B)}{A \land \neg B}$$

$\exists p.$ $($$Person(p)$ $\land$

$\quad \neg\exists q.$ $(Person(q) \land p \neq q \land$

$\quad\quad Loves(p, q)$

$\quad)$

$)$

$$\frac{\neg(A \to B)}{A \land \neg B}$$

$$
\begin{aligned}
&\exists p.\ (Person(p) \land \\
&\quad \neg\exists q.\ (Person(q) \land p \neq q \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\begin{array}{l}
\exists p.\ (Person(p)\ \land \\
\quad \neg\exists q.\ (Person(q) \land p \neq q\ \land \\
\qquad Loves(p, q) \\
\quad) \\
)
\end{array}
$$

$$\boxed{\dfrac{\neg\exists x.\ A}{\forall x.\ \neg A}}$$

$$
\begin{aligned}
&\exists p.\ (Person(p)\ \land \\
&\quad \neg\exists q.\ (Person(q)\ \land\ p \neq q\ \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$\boxed{\dfrac{\neg\exists x.\ A}{\forall x.\ \neg A}}$$

$$
\begin{aligned}
&\exists p.\ (Person(p) \land \\
&\quad \neg\exists q.\ (Person(q) \land p \neq q \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$\frac{\neg\exists x.\ A}{\forall x.\ \neg A}$$

$$
\begin{aligned}
&\exists p.\ (Person(p)\ \land \\
&\quad \forall q.\ \neg(Person(q)\ \land\ p \neq q\ \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$\boxed{\dfrac{\neg\exists x.\ A}{\forall x.\ \neg A}}$$

$$
\begin{aligned}
&\exists p.\ (\mathit{Person}(p) \land \\
&\quad \forall q.\ \neg(\mathit{Person}(q) \land p \neq q \land \\
&\qquad \mathit{Loves}(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\exists p.\ (\mathit{Person}(p)\ \land \\
&\quad \forall q.\ \neg(\mathit{Person}(q) \land p \neq q\ \land \\
&\qquad \mathit{Loves}(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$\frac{\neg(A \land B)}{A \to \neg B}$$

$$
\begin{array}{l}
\exists p.\ (\mathit{Person}(p)\ \land \\
\quad \forall q.\ \neg(\mathit{Person}(q) \land p \neq q\ \land \\
\qquad \mathit{Loves}(p, q)) \\
\quad) \\
)
\end{array}
$$

$$
\boxed{\dfrac{\neg({\color{red}A} \land {\color{blue}B})}{{\color{red}A} \to \neg {\color{blue}B}}}
$$

$$
\begin{aligned}
&\exists p.\ (Person(p) \land \\
&\quad \forall q.\ \neg(Person(q) \land p \neq q \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$\frac{\neg(A \land B)}{A \to \neg B}$$

$$
\begin{aligned}
&\exists p.\ (Person(p) \land \\
&\quad \forall q.\ (Person(q) \land p \neq q \to \\
&\qquad \neg Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\frac{\neg(A \land B)}{A \to \neg B}
$$

$$
\begin{array}{l}
\exists p.\ (Person(p)\ \land \\
\quad \forall q.\ (Person(q)\ \land\ p \neq q \rightarrow \\
\quad\quad \neg Loves(p, q) \\
\quad) \\
)
\end{array}
$$

$$
\begin{aligned}
&\forall p.\ (\mathit{Person}(p) \to \\
&\quad \exists q.\ (\mathit{Person}(q) \land p \neq q \land \\
&\qquad \mathit{Loves}(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\exists p.\ (\mathit{Person}(p) \land \\
&\quad \forall q.\ (\mathit{Person}(q) \land p \neq q \to \\
&\qquad \neg \mathit{Loves}(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\exists p.\ (Person(p) \land \\
&\quad \forall q.\ (Person(q) \land p \neq q \rightarrow \\
&\qquad Loves(q, p) \\
&\quad) \\
&)
\end{aligned}
$$

Your turn!

Try negating this formula with the other folks at your table. See what you come up with!

$$
\begin{aligned}
&\neg\exists p.\ (Person(p) \land \\
&\quad \forall q.\ (Person(q) \land p \neq q \rightarrow \\
&\qquad Loves(q, p) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\forall p.\ \neg(Person(p)\ \land \\
&\quad \forall q.\ (Person(q) \land p \neq q \rightarrow \\
&\qquad Loves(q, p) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\forall p.\ (\mathit{Person}(p) \to \\
&\quad \neg\forall q.\ (\mathit{Person}(q) \land p \neq q \to \\
&\qquad \mathit{Loves}(q, p) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\begin{array}{l}
\forall p.\ (Person(p) \to \\
\quad \exists q.\ \neg(Person(q) \land p \neq q \to \\
\quad\quad Loves(q, p) \\
\quad) \\
)
\end{array}
$$

$$
\begin{array}{l}
\forall p.\ (Person(p) \rightarrow \\
\quad \exists q.\ (Person(q) \land p \neq q \land \\
\quad\quad \neg Loves(q, p) \\
\quad) \\
)
\end{array}
$$

$$
\begin{array}{l}
\exists p.\ (Person(p)\ \land \\
\quad \forall q.\ (Person(q) \land p \neq q \rightarrow \\
\qquad Loves(q, p) \\
\quad) \\
)
\end{array}
$$

$$
\begin{array}{l}
\forall p.\ (Person(p) \rightarrow \\
\quad \exists q.\ (Person(q) \land p \neq q\ \land \\
\qquad \neg Loves(q, p) \\
\quad) \\
)
\end{array}
$$

Techniques: ***Translating Statements***

Common Patterns

- A statement of the form

  $$\forall x.\ (P(x) \to Q(x))$$

  can be read as "all *P*'s are *Q*'s."
- A statement of the form

  $$\exists x.\ (P(x) \land Q(x))$$

  can be read as "there is a *P* that is also a *Q*" or "some *P*'s are *Q*'s."
- ***Remember:*** If you see ∃ paired with → or ∀ paired with ∧, the statement is probably incorrect!

Given the predicates

- *Person*(*p*), which states that *p* is a person, and
- *CanLearnFrom*(*x, y*), which says that *x* can learn from *y*,

write a statement in first-order logic that says “everyone has someone they can learn from.”

Everyone has someone they can learn from

Every person p has someone they can learn from

Every person p has someone they can learn from

“All *A*s are *B*s.”

$$\forall x.\ (A(x) \rightarrow B(x))$$

$\forall p.\ (Person(p) \rightarrow$

p has someone they can learn from

$)$

$\forall p.\ (Person(p) \rightarrow$

there is a person q that p can learn from

$)$

$\forall p.\ (Person(p) \rightarrow$

there is a person q that p can learn from

$)$

"Some *A*s are *B*s."

$\boldsymbol{\exists x.\ (A(x) \land B(x))}$

$$
\begin{aligned}
&\forall p.\ (Person(p) \rightarrow \\
&\quad \exists q.\ (Person(q) \land \\
&\qquad \textit{p can learn from q} \\
&\quad) \\
&)
\end{aligned}
$$

$$
\begin{array}{l}
\forall p.\ (\mathit{Person}(p) \to \\
\qquad \exists q.\ (\mathit{Person}(q) \land \\
\qquad\qquad \mathit{CanLearnFrom}(p, q) \\
\qquad) \\
)
\end{array}
$$

$$
\begin{aligned}
&\forall p.\ (Person(p) \to \\
&\quad \exists q.\ (Person(q)\ \land \\
&\quad\quad CanLearnFrom(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

- ☐ ***Pair quantifiers with the appropriate connectives.***
- ☐ ***Use whitespace and indentation to clarify meaning.***
- ☐ ***Defer quantifiers until they're needed.***
- ☐ ***Check the types of all terms in the formula.***
- ☐ ***Check the scoping on each variable.***

Consider this statement:

"If someone is happy, then everyone is happy."

What is the *contrapositive* of this statement?

If someone is happy, then everyone is happy *someone is happy* → *everyone is happy* *someone is happy* $\rightarrow (\forall x.\ Happy(x))$

$$(\exists x.\ Happy(x)) \rightarrow (\forall x.\ Happy(x))$$

$$(\exists x.\ \mathit{Happy}(x)) \rightarrow (\forall x.\ \mathit{Happy}(x))$$

$$(\exists x.\ Happy(x)) \rightarrow (\forall x.\ Happy(x))$$

$$\neg(\forall x.\ Happy(x)) \rightarrow \neg(\exists x.\ Happy(x))$$

$$(\exists x.\ \neg Happy(x)) \rightarrow \neg(\exists x.\ Happy(x))$$

$$(\exists x.\ \neg Happy(x)) \rightarrow (\forall x.\ \neg Happy(x))$$

CONTRAPOSITIVE of "If someone is happy, then everyone is happy"

$$(\exists x.\ \neg Happy(x)) \rightarrow (\forall x.\ \neg Happy(x))$$

"If someone is not happy, then everyone is not happy."

Consider this statement:

“If someone is happy, then everyone is happy.”

What is the *negation* of this statement?

$$(\exists x.\ \mathit{Happy}(x)) \rightarrow (\forall x.\ \mathit{Happy}(x))$$

$$(\exists x.\ Happy(x)) \to (\forall x.\ Happy(x))$$

$$\neg((\exists x.\ Happy(x)) \to (\forall x.\ Happy(x)))$$

$$(\exists x.\ \mathit{Happy}(x)) \to (\forall x.\ \mathit{Happy}(x))$$

$$(\exists x.\ \mathit{Happy}(x)) \land \neg(\forall x.\ \mathit{Happy}(x))$$

$$(\exists x.\ Happy(x)) \rightarrow (\forall x.\ Happy(x))$$

$$(\exists x.\ Happy(x)) \land (\exists x.\ \neg Happy(x))$$

"If someone is happy, then everyone is happy"

$$(\exists x.\ Happy(x)) \rightarrow (\forall x.\ Happy(x))$$

$$(\exists x.\ Happy(x)) \land (\exists x.\ \neg Happy(x))$$

"Someone is happy and someone is not happy."