Exercise 1 (25 points). Formulate each of the below as a single statement (proposition or predicate), using only mathematical and logical notation that has been defined in class. For example, the use of logical quantifiers and connectives, and arithmetic, number-theoretic, and set-theoretic operations is allowed, as is the use of sets like \( \mathbb{Q}, \mathbb{R} \), etc., but not the use of English-language words.

(a) Every positive integer is either even or odd.
(b) \( p \) is prime.
(c) There are infinitely many primes.
(d) If \( a \) and \( b \) are integers and \( b \neq 0 \), then there is a unique pair of integers \( q \) and \( r \), such that \( a = qb + r \) and \( 0 \leq r < |b| \).
(e) If \( a \) and \( n \) are coprime then there exists exactly one \( x \in \mathbb{Z}_n \) for which \( ax \equiv b \mod n \), for any \( b \in \mathbb{Z} \).

Exercise 2 (25 points). After completing the previous exercise, write the negation of each of your logical statements, such that the symbol \( \neg \) does not appear in your statements. (That is, eliminate negated quantifiers and negated compounds as you have learned in class, and then replace statements such as \( \neg(a \mid b) \) by statements like \( a/b \).) Read the negations out in natural language and check for yourself that you understand why these are the right negations for the statements in the previous exercise.

Exercise 3 (25 points). Which of the following are valid equivalences? Prove.

(a) \( (P \rightarrow Q) \land (P \rightarrow R) \equiv (P \rightarrow (Q \land R)) \)
(b) \( (P \rightarrow R) \land (Q \rightarrow R) \equiv ((P \land Q) \rightarrow R) \)
(c) \( (P \land Q) \lor R \equiv (P \lor R) \land (Q \lor R) \)
(d) \( (P \lor Q) \land R \equiv (P \land R) \lor (Q \land R) \)

Exercise 4 (25 points). Define a new connective \( \nand \) (read “nand”) as follows:

\[
\begin{array}{ccc}
P & Q & P \nand Q \\
T & T & F \\
T & F & T \\
F & T & T \\
F & F & T \\
\end{array}
\]

Show that the propositions \( \neg P, P \land Q, \) and \( P \lor Q \) can be expressed in terms of \( \nand \) alone, with no other connectives. Conclude that \( \nand \) is universal, meaning that a proposition that involves any of the connectives we have seen so far can be expressed using only \( \nand \).