CS 103X: Discrete Structures
Homework Assignment 1

Due January 18, 2007

Exercise 1 (5 points)**.** If $2^A \subseteq 2^B$, what is the relation between A and B?

Exercise 2 (5 points)**.** Prove or give a counterexample: If $A \subset B$ and $A \subset C$, then $A \subset B \cap C$.

Exercise 3 (10 points)**.** Prove or give a counterexample for each of the following:

(a) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

(b) If $A \in B$ and $B \in C$, then $A \in C$.

Exercise 4 (20 points)**.** Let A be a set with m elements and B be a set with n elements, and assume $m < n$. For each of the following sets, give upper and lower bounds on their cardinality and provide sufficient conditions for each bound.

(a) $A \cap B$

(b) $A \cup B$

(c) $A \backslash B$

(d) $2^A \cup A$

Exercise 5 (15 points)**.** If $a(t)$, $b(t)$ and $c(t)$ are the lengths of the three sides of a triangle t in *non-decreasing order* (i.e. $a(t) \leq b(t) \leq c(t)$), we define the sets:

- $X := \{\text{Triangle } t : a(t) = b(t)\}$
- $Y := \{\text{Triangle } t : b(t) = c(t)\}$
- $T :=$ the set of all triangles

Using only set operations on these three sets, define:

(a) The set of all equilateral triangles (all sides equal)

(b) The set of all isosceles triangles (at least two sides equal)

(c) The set of all scalene triangles (no two sides equal)

Exercise 6 (15 points)**.** Is it possible for every member of a set A to also be a subset of A? If so, is it possible for all cardinalities? Provide positive examples or proofs as to why this cannot be.

Exercise 7 (30 points)**.** In addition to union ($\cup$), intersection ($\cap$), difference ($\setminus$) and power set (2^A), let us add the following two operations to our dealings with sets:

- Pairwise addition: $A \oplus B := \{a + b : a \in A, b \in B\}$ (This is also called the Minkowski addition of sets A and B.)
- Pairwise multiplication: $A \otimes B := \{a \times b : a \in A, b \in B\}$

For example, if A is $\{1, 2\}$ and B is $\{10, 100\}$, then $A \oplus B = \{11, 12, 101, 102\}$ and $A \otimes B = \{10, 20, 100, 200\}$. Now answer the following questions:

(a) (10 points) Succinctly describe the following sets:

  i. $\mathbb{N} \oplus \emptyset$

  ii. $\mathbb{N} \oplus \mathbb{N}$

  iii. $\mathbb{N}^+ \oplus \mathbb{N}^+$

  iv. $\mathbb{N}^+ \otimes \mathbb{N}^+$

(b) (10 points) If E is the set of all positive even numbers, what's the shortest way to write the set of all positive multiples of 4? Of 8?

(c) (10 points) Let $S := \{n^2 : n \in \mathbb{N}^+\}$. A *Pythagorean triple* consists of three positive integers x, y and z such that $x^2 + y^2 = z^2$. Construct the set of all possible z that could appear as the last element of a Pythagorean triple using only the set S and the set operations we have so far.

(d) [**Optional, attempt only after solving all the other exercises**] A *prime number* is an integer greater than 1 that has 1 and itself as its only positive divisors. The first few prime numbers are $2, 3, 5, 7, 11, 13, \ldots$. Let P be the set of all odd prime numbers (2 is the only even prime). What can we say about the set $P \oplus P$?