Exercise 1 (10 points). The complement of a graph $G = (V, E)$ is the graph

$$(V, \{\{x, y\} : x, y \in E, x \neq y\} \setminus E).$$

A graph is self-complementary if it is isomorphic to its complement.

(a) Prove that no simple graph with two or three vertices is self-complementary, without enumerating all isomorphisms of such graphs.

(b) Find examples of self-complementary simple graphs with 4 and 5 vertices.

Exercise 2 (10 points). Prove that if a graph has at most $m$ vertices of degree at most $n$ and all other vertices have degree at most $k$, with $k < n$ and $m < n$, then the graph is colorable with $m + k + 1$ colors.

Exercise 3 (30 points). Prove or disprove, for a graph $G$ on a finite set of $n$ vertices:

(a) If every vertex of $G$ has degree 2, then $G$ contains a cycle.

(b) If $G$ is disconnected, then its complement is connected.

(c) If $T$ is a non-cyclic tour in $G$, and no strictly longer tour in $G$ contains $T$, then both endpoints of $T$ have odd degree.

Exercise 4 (15 points). Consider $m$ graphs $G_1 = (V_1, E_1), G_2 = (V_2, E_2), \ldots, G_m = (V_m, E_m)$. Their union can be defined as

$$\bigcup_{i=1}^{m} G_i = \left(\bigcup_{i=1}^{m} V_i, \bigcup_{i=1}^{m} E_i\right).$$

Show that, for any natural number $n \geq 2$, the clique $K_n$ can be expressed as the union of $k$ bipartite graphs if $n \leq 2^k$.

Exercise 5 (15 points). A binary tree is defined inductively as follows:

- A single vertex $u$ defines a binary tree with root $u$.
- A vertex $u$ linked by edges to the roots of one or two binary trees defines a binary tree with root $u$.

The following figure illustrates the three possibilities:
$T_1$ and $T_2$ are called *subtrees*, $u$ is the *parent* of the roots of the subtrees, and these roots are *children* of $u$. The vertices of a binary tree without any children are called *leaves*. Here's an example of a binary tree:

![Binary Tree Example](image)

The *distance* between two vertices of a tree is the number of edges in the shortest path connecting them. The *height* of the tree is the maximum distance between the root and a leaf. Prove that the height of a binary tree with $n$ vertices is at least $\log_2 n$. (Hint: Strong induction.)

**Exercise 6** (20 points). Given a graph $G = (V, E)$, an edge $e \in E$ is said to be a bridge if the graph $G' = (V, E \setminus \{e\})$ has more connected components than $G$. Let $G$ be a bipartite $k$-regular graph for $k \geq 2$. Prove that $G$ has no bridge.