CS 103X: Discrete Structures
Homework Assignment 1

Due January 18, 2007

Exercise 1 (5 points). If $2^A \subseteq 2^B$, what is the relation between A and B?

Solution Recall that 2^A is the set of all subsets of A, including A itself. The condition tells us that every subset of A is also a subset of B, and in particular A itself is a subset of B. So $A \subseteq B$.

Exercise 2 (5 points). Prove or give a counterexample: If $A \subset B$ and $A \subset C$, then $A \subset B \cap C$.

Solution Since $\subset$ denotes the "proper subset" relation, this will not hold whenever $A = B \cap C$. E.g. take $A = \{1, 2\}$, $B = \{1, 2, 3, 4\}$, $C = \{1, 2, 5, 6\}$.

Exercise 3 (10 points). Prove or give a counterexample for each of the following:

(a) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

(b) If $A \in B$ and $B \in C$, then $A \in C$.

Solution

(a) Consider any element $a \in A$. Since $A \subseteq B$, every element of A is also an element of B, so $a \in B$. By the same reasoning, $a \in C$ since $B \subseteq C$. Thus every element of A is an element of C, so $A \subseteq C$.

(b) Let $A = \{1\}$, $B = \{1, \{1\}\}$, and $C = \{\{1, \{1\}\}, 3, \{4\}\}$. These sets satisfy $A \in B$ and $B \in C$, but $A \notin C$.

Exercise 4 (20 points). Let A be a set with m elements and B be a set with n elements, and assume $m < n$. For each of the following sets, give upper and lower bounds on their cardinality and provide sufficient conditions for each bound to hold with equality.

(a) $A \cap B$

(b) $A \cup B$

(c) $A \backslash B$

(d) $2^A \cup A$

Solution

(a) Upper bound: The largest intersection has all of the members of the smaller set, so m is the upper bound. This will be satisfied if $A \subset B$. Lower bound: The intersection can be empty, so 0 is the lower bound. We say A and B are *disjoint* if $A \cap B = \phi$. Using our current notation, this can be expressed as $A \backslash B = A$ or equivalently $B \backslash A = B$.

(b) Upper bound: The largest union occurs if all members of both sets are in the union, so the sets must be disjoint. So the bound is $m + n$ elements. Lower bound: The smallest union occurs when the two sets have a maximal number of common elements, this occurs when $A \subset B$. In this case the size of the resulting set is n.

(c) Upper bound: This will occur when A and B have no elements in common, meaning that they are disjoint. Then $A \backslash B = A$ so the bound is m. Lower bound: Like the union, this will be smallest when there are a maximal number of common elements, so $A \subset B$. Then $A \backslash B = \phi$ and the bound is 0 (Remember that A is the smaller set; the lower bound on $B \backslash A$ would be $n - m$).

(d) Upper bound: If 2^A and A are disjoint, their union will have $m + 2^m$ elements. Recall that all members of a power set are themselves sets, so any set A with no members that are sets will be disjoint with its power set. Lower bound: If $A \subseteq 2^A$, then the union will have 2^m elements. Since the power set is the set of all subsets, this occurs for a set for which all members are also subsets of a set. See the solution to Exercise 6 for more details.

Exercise 5 (15 points)**.** If $a(t)$, $b(t)$ and $c(t)$ are the lengths of the three sides of a triangle t in *non-decreasing order* (i.e. $a(t) \leq b(t) \leq c(t)$), we define the sets:

- $X := \{\text{Triangle } t : a(t) = b(t)\}$
- $Y := \{\text{Triangle } t : b(t) = c(t)\}$
- $T :=$ the set of all triangles

Using only set operations on these three sets, define:

(a) The set of all equilateral triangles (all sides equal)

(b) The set of all isosceles triangles (at least two sides equal)

(c) The set of all scalene triangles (no two sides equal)

Solution

(a) We require $a(t) = b(t)$ and $b(t) = c(t)$ (this obviously implies $a(t) = c(t)$), so the set is $X \cap Y$.

(b) An isosceles triangle t can have

i. $a(t) = b(t)$, or

ii. $b(t) = c(t)$, or

iii. $a(t) = c(t)$.

Now we've assumed that $a(t)$, $b(t)$ and $c(t)$ are in non-decreasing order, so the last condition holds if and only if both the first two do. So the required set is $X \cup Y \cup (X \cap Y)$, which simplifies to just $X \cup Y$.

(c) A scalene triangle has its two smaller sides $a(t)$ and $b(t)$ unequal (set $T \backslash X$) and its two larger sides $b(t)$ and $c(t)$ unequal (set $T \backslash Y$). Since the sides are listed in non-decreasing order, either of the above conditions guarantees $a(t) \neq c(t)$. So the required set is $(T \backslash X) \cap (T \backslash Y)$.

An alternative argument is: A triangle is scalene if and only if it is not isosceles. So using the result of the previous part, the set of scalene triangles is $T \backslash (X \cup Y)$. It's easy to confirm that the answers given by the two arguments are actually the same – this is an instance of a general rule called De Morgan's Law.

Exercise 6 (15 points)**.** Is it possible for every member of a set A to also be a subset of A? If so, is it possible for all cardinalities? Provide positive examples or proofs as to why this cannot be.

Solution It is possible for all cardinalities. Define the sequence $a_0 = \phi$, $a_1 = \{\phi\}$, $a_2 = \{\{\phi\}\}$, etc. For any nonnegative integer n, the set $A = \{a_0, a_1, ..., a_{n-1}\}$ satisfies the conditions — specifically, the member a_i is also the set $\{a_{i-1}\}$. Note that for $n = 0$, $A = \phi$ which also works; since ϕ has no members it will satisfy any condition that states a property for all members of a set.

Exercise 7 (30 points)**.** In addition to union ($\cup$), intersection ($\cap$), difference ($\backslash$) and power set (2^A), let us add the following two operations to our dealings with sets:

- Pairwise addition: $A \oplus B := \{a + b : a \in A, b \in B\}$ (This is also called the Minkowski addition of sets A and B.)
- Pairwise multiplication: $A \otimes B := \{a \times b : a \in A, b \in B\}$

For example, if A is $\{1, 2\}$ and B is $\{10, 100\}$, then $A \oplus B = \{11, 12, 101, 102\}$ and $A \otimes B = \{10, 20, 100, 200\}$. Now answer the following questions:

(a) (10 points) Succinctly describe the following sets:

i. $\mathbb{N} \oplus \emptyset$

ii. $\mathbb{N} \oplus \mathbb{N}$

iii. $\mathbb{N}^+ \oplus \mathbb{N}^+$

iv. $\mathbb{N}^+ \otimes \mathbb{N}^+$

(b) (10 points) If E is the set of all positive even numbers, what's the shortest way to write the set of all positive multiples of 4? Of 8?

(c) (10 points) Let $S := \{n^2 : n \in \mathbb{N}^+\}$. A *Pythagorean triple* consists of three positive integers x, y and z such that $x^2 + y^2 = z^2$. Construct the set of all possible z^2 such that z is the last element of a Pythagorean triple using only the set S and the set operations we have so far.

(d) [**Optional, attempt only after solving all the other exercises**] A *prime number* is an integer greater than 1 that has 1 and itself as its only positive divisors. The first few prime numbers are $2, 3, 5, 7, 11, 13, \ldots$. Let P be the set of all odd prime numbers (2 is the only even prime). What can we say about the set $P \oplus P$?

Solution You should first convince yourself that if $C \subseteq A$, then $C \oplus B \subseteq A \oplus B$, and similarly for $\otimes$.

(a) i. $\emptyset$

ii. $\mathbb{N}$. The sum of any two natural numbers is a natural number, so $\mathbb{N} \oplus \mathbb{N} \subseteq \mathbb{N}$, and since $0 \in \mathbb{N}$, $\mathbb{N} = \mathbb{N} \oplus \{0\} \subseteq \mathbb{N} \oplus \mathbb{N}$. So $\mathbb{N} \oplus \mathbb{N} = \mathbb{N}$.

iii. $\mathbb{N}^+ \setminus \{1\}$. The sum of two positive integers is an integer ≥ 2, and every integer $n \geq 2$ can be written as $(n-1)+1$, where we note that $n-1$ and 1 are both positive integers.

iv. $\mathbb{N}^+$. Exactly the same argument as the second part, replacing $\mathbb{N}$ with $\mathbb{N}^+$, $\oplus$ with $\otimes$ and 0 with 1.

(b) Let F be the set of multiples of 4. We claim that $F = E \otimes E$. Every positive even number can be written as $2k$ for some $k \in \mathbb{N}^+$, so $E \otimes E$ consists of elements of the general form $2j \times 2k = 4jk$, for $j, k \in \mathbb{N}^+$. In other words, every element of $E \otimes E$ is a multiple of 4, so $E \otimes E \subseteq F$. Also, every multiple of 4 is of the form $4k = 2 \times 2k$, for $k \in \mathbb{N}^+$, so $F \subseteq \{2\} \otimes E \subseteq E \otimes E$. This proves the claim.

A virtually identical argument shows that T, the set of positive multiples of 8, is $E \otimes E \otimes E$.

(c) Observe that the set of all possible numbers of the form $x^2 + y^2$, where x and y are positive integers, is $S \oplus S$. If such a number is also the square of a positive integer z, it must be in $(S \oplus S) \cap S$, which is the required set.

Note: There was an error in the original version of the homework which asked for the set of z and not z^2. The original problem cannot be solved as asked.

(d) `http://en.wikipedia.org/wiki/Goldbach_conjecture`