CS103X: Discrete Structures
Final Review

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Exercise 1. Let S be some set and T be a subset of S. For subsets A and B of S, say $A \sim B$ if and only if $(A \cup B) \setminus (A \cap B) \subseteq T$.

Solution $(A \cup A) \setminus (A \cap A) = \emptyset \subseteq T$, so $A \sim A$ and the relation is reflexive. If $A \sim B$, then $(A \cup B) \setminus (A \cap B) \subseteq T$, but since $\cup$ and $\cap$ are symmetric, $A \cup B = B \cup A$ and $A \cap B = B \cap A$, so $(B \cup A) \setminus (B \cap A) \subseteq T$. So $B \sim A$ and the relation is symmetric. Assume $A \sim B$ and $B \sim C$. It it easy to prove that e is an element of $S = (A \cup B) \setminus (A \cap B)$ if and only if it is in exactly one of A and B. (If it is in exactly one, then it is in $A \cup B$ but not in $A \cap B$ and hence is preserved by the set subtraction. If it is in neither, then it is not in $A \cup B$ and hence not in S, and if it is in both then it is removed from S by subtracting $A \cap B$.) So $A \sim B$ implies that every such element is in T. Similarly $B \sim C$ implies that every element in exactly one of B and C is in T. Now consider an element e in exactly one of A and C. Assume it is in A, hence not in C. If it is also in B, then it satisfies the condition to be an element of $(B \cup C) \setminus (B \cap C)$ and hence is in T. If e is not in B, then it satisfies the condition to be in $(A \cup B) \setminus (A \cap B)$ and hence is in T. An analogous line of reasoning applies to show that if e is in C but not in A then it is in T. So $A \sim C$ and the relation is transitive. Hence $\sim$ is an equivalence.

Exercise 2. Consider m graphs $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$, $\ldots$, $G_m = (V_m, E_m)$. Their union can be defined as

$$\bigcup_{i=1}^{m} G_i = \left(\bigcup_{i=1}^{m} V_i, \bigcup_{i=1}^{m} E_i \right).$$

Show that, for any natural number $n \geq 2$, the clique K_n can be expressed as the union of k bipartite graphs if $n \leq 2^k$.

Solution We proceed by induction on k. For $k = 1$, there are two cliques: K_1 is just a single point, which is trivially a bipartite graph. K_2 is also a single bipartite graph (each vertex in its own group). Now, for the inductive step assume the claim holds for $k = m$. Now for any $n \leq 2^{m+1}$, let $a = \lfloor \frac{n}{2} \rfloor$ and $b = \lceil \frac{n}{2} \rceil$, and divide the vertices of K_n into disjoint sets A, B with $|A| = a$ and $|B| = b$. We can define a bipartite graph G with vertices $A \cup B$ and edges v, w with $v \in A, w \in B$. Removing all of these edges from K_n leave two cliques K_a and K_b. Since $a, b \leq 2^m$, each of these cliques can be represented as a union of m bipartite graphs. Since the two cliques have disjoint vertex sets, we can say that the union of a bipartite graph over the vertices of K_a and a bipartite graph over the vertices of K_b will still be a bipartite graph. Thus the two cliques together can be represented as the union of m bipartite graphs, and adding G to the union represents all of K_n as $m + 1$ bipartite graphs. This completes the inductive step and thus by induction the property holds for all k.

Exercise 3. The drama club has m members and the dance club has n members. For an upcoming musical, a committee of k people needs to be formed with at least one member from each club. If the clubs have exactly r members in common, what is the number of ways the committee may be chosen? Substantiate.

Solution There are $m+n-r$ total people to choose from, so without the restriction the number of ways is $\binom{m+n-r}{k}$. Then we subtract the ways that won't work, which is when no people from one club are chosen. There are $m-r$ only in the dance club and $n-r$ only in the drama club. Thus there are $\binom{m-r}{k}$ ways to choose while having no one from the drama club chosen, and $\binom{n-r}{k}$ ways to pick no one from the dance club. Subtracting these gives a final answer of $\binom{m+n-r}{k} - \binom{m-r}{k} - \binom{n-r}{k}$.

Exercise 4. You already know from Bezout's Identity that if a and b are coprime integers, then there are integers x and y such that $ax + by = 1$. Now prove the same result using the Pigeonhole Principle. (You may assume that a and b are positive.)
Hint: Take the remainders, modulo b, of the first $b-1$ positive multiples of a, and consider what happens if 1 is not in this set.

Solution a and b are coprime, so at most one of a and b can be 1. Without loss of generality assume $b \neq 1$ — this ensures $b \nmid a$. We can rewrite $ax + by = 1$ as $ax = (-y)b + 1$. This suggests that we consider the remainders of multiples of a modulo b, i.e. the integers $a \text{ rem } b, 2a \text{ rem } b, 3a \text{ rem } b, \ldots, (b-1)a \text{ rem } b$. Assume, for the sake of contradiction, that none of them is 1. Then, since there are $b-1$ of them and they all lie in the set $\{2, 3, \ldots, b-1\}$ (0 is absent since $b \nmid a$), which has $b-2$ elements, the Pigeonhole Principle tells us that two of them must be equal. Say $pa \text{ rem } b = qa \text{ rem } b$, for $1 \leq p, q < b$. This implies $pa \equiv_b qa$, or $(p-q)a \equiv_b 0$. But since a and b are coprime, this means $p-q$ is a multiple of b, which is impossible since p and q are unequal positive integers less than b (so $0 < |p-q| < b$). Hence we have reached a contradiction and 1 must be one of the remainders, say $pa \text{ rem } b = 1$. Then $pa = qb + 1$ for some q, and choosing $x = p$, $y = -q$ we get the required result.

Exercise 5. How many nonnegative integers less than or equal to 300 are coprime with 144? Substantiate.

Solution 144 has a prime factorization of all 2's and 3's. So, by inclusion-exclusion, the answer is 300 − (number divisible by 2) − (number divisible by 3) + (number divisible by 2 and 3). Of course, the last is the same as the number divisible by 6. Since 300 is divisible by 2,3, and 6, the formula is $300 - \frac{300}{2} - \frac{300}{3} + \frac{300}{6} = 300 - 150 - 100 + 50 = 100$.

Exercise 6. A certain company wants to have security for their computer systems. So they have given everyone a name and password. A length 10 word containing each of the characters: $a, d, e, f, i, l, o, p, r, s$, is called a cword. A password will be a cword which does not contain any of the subwords fails, failed, or drop.Use the Inclusion-exclusion Principle to find a simple formula for the number of passwords.

Solution There are 7!cwords that contain drop, 6!that contain fails, and 5!that contain failed. There are 3!cwords containing both drop and fails. No cword can contain both fails and failed. The cwords containing both drop and failed come from taking the subword failedrop and the remaining letter s in any order, so there are 2!of them. So by Inclusion-exclusion,we have the number of cwords containing at least one of the three forbidden subwords is

$$(7! + 6! + 5!) - (3! + 0 + 2!) + 0 = 5!(48) - 8$$

Among the 10!cwords, the remaining ones are passwords, so the number of passwords is

$$10! - 5!(48) + 8 = 3,623,048$$

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