CS103X: Discrete Structures
Homework Assignment 2

Due February 1, 2008

Exercise 1 (10 Points). Prove or give a counterexample for the following:

Use the Fundamental Theorem of Arithmetic to prove that for $n \in \mathbb{N}$, $\sqrt{n}$ is irrational unless n is a perfect square, that is, unless there exists $a \in \mathbb{N}$ for which $n = a^2$.

Exercise 2 (20 Points). Prove or disprove, for integers a, b, c and d:

(a) If $a|b$ and $a|c$, then $a|(b + c)$.

(b) If $a|bc$ and $gcd(a, b) = 1$, then $a|c$.

(c) If a and b are perfect squares and $a|b$, then $\sqrt{a}|\sqrt{b}$.

(d) If $ab|cd$, then $a|c$ or $a|d$.

Exercise 3 (25 Points). On Euclids algorithm:

(a) Write the algorithm in pseudo-code. (10 points)

(b) Prove that Euclids Algorithm correctly finds the GCD of a and b in a finite number of steps. (10 points)

(c) Use the algorithm to calculate gcd(1247, 899). Write out the complete sequence of derivations. (5 points)

Exercise 4 (20 Points) Some prime facts:

(a) Prove that for every positive integer n, there exist at least n consecutive composite numbers. (10 points)

(b) Prove that if an integer $n \geq 2$ is such that there is no prime $p \leq \sqrt{n}$ that divides n, then n is a prime. (10 points)

Exercise 5 (25 Points) A fun game:

To start with, there is a chart with numbers 1211 and 1729 written on it. Now you and I take turns and you go first. On each players turn, he or she must write a new positive integer on the board that is the difference of two numbers that are already there. The first person who cannot create a new number loses the game.
For example, your first move must be $1729 - 1211 = 518$. Then I could play either $1211 - 518 = 693$ or $1729 - 518 = 1211$, and so forth.

(a) Prove every number written on the chart is a multiple of 7 less than or equal to 1729. (10 points)

(b) Prove that every positive multiple of 7 less than or equal to 1729 is on the chart at the end of the game. (10 points)

(c) Can you predict the winner? What if I go first? (5 points)