Exercise 1 (10 points). Silicon Valley questions:

(a) How many possible six-figure salaries (in whole dollar amounts) are there that contain at least three distinct digits?

(b) Second Silicon Valley question: What is the number of six-figure salaries that are not multiples of either 3, 5, or 7.

Exercise 2 (15 points). A rook on a chessboard is said to put another chess piece under attack if they are in the same row or column.

(a) How many ways are there to arrange 8 rooks on a chessboard (the usual 8 × 8 one) so that none are under attack?

(b) How many ways are there to arrange \( k \) rooks on an \( n \times n \) chessboard so that none are under attack?

(c) Imagine a three-dimensional chess variant played on a 8 × 8 × 8 board. (512 cells overall.) Call it Weir-D Chess. A battleship is a Weir-D Chess piece that can attack any piece that is in the same two-dimensional layer, along some coordinate. (For example, a battleship in position (5,2,6) puts cell (8,2,1) under attack, but not cell (8,3,1).) How many ways are there to arrange 8 battleships on a Weir-D Chess board so that none are under attack?

Give solutions with no summation.

Exercise 3 (15 points). A function \( f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, m\} \) is called monotone nondecreasing if \( 1 \leq i < j \leq n \Rightarrow f(i) \leq f(j) \).

(a) How many such functions are there?

(b) How many such functions are there that are surjective?

(c) How many such functions are there that are injective?

Exercise 4 (10 points). How many ways are there to express a positive integer \( n \) as:

(a) A sum of \( k \) natural numbers? (For example, if \( n = 2 \) and \( k = 3 \) the answer is 6, since \( 2 = 2 + 0 + 0 = 0 + 2 + 0 = 0 + 0 + 2 = 1 + 1 + 0 = 1 + 0 + 1 = 0 + 1 + 1 \).)

(b) A sum of positive integers?

The order of the summands is important. (Imagine the summation written down.)

Exercise 5 (10 points). Prove either algebraically or combinatorially:

(a) For \( p, n \geq 0 \), \( \sum_{k=p}^{n} \binom{k}{p} = \binom{n+1}{p+1} \)

(b) \( \sum_{k=0}^{n} \binom{m+k}{k} = \binom{m+n+1}{n} \)
Exercise 6 (10 points). Give a closed-form expression (without summation) for the following:

\[
\sum_{k=0}^{n} 2^k \binom{n}{k}.
\]

Exercise 7 (10 points). In a mathematics contest with three problems, 80% of the participants solved the first problem, 75% solved the second and 70% solved the third. Prove that at least 25% of the participants solved all three problems. (The claim might seem obvious — find a proof.)

Exercise 8 (10 points). What is the number of integer solutions of the equation

\[x_1 + x_2 + x_3 = 50,\]

such that \(0 \leq x_i \leq 20\) for each \(1 \leq i \leq 3\)?

Exercise 9 (10 points). There are \(n\) people at a party, and each person has arrived in a different hat. The revelry leaves them slightly tipsy, so each of them goes home wearing someone else’s hat. Find the number of ways of putting \(n\) hats on \(n\) people so that no person is wearing his/her own hat. Give the full proof.