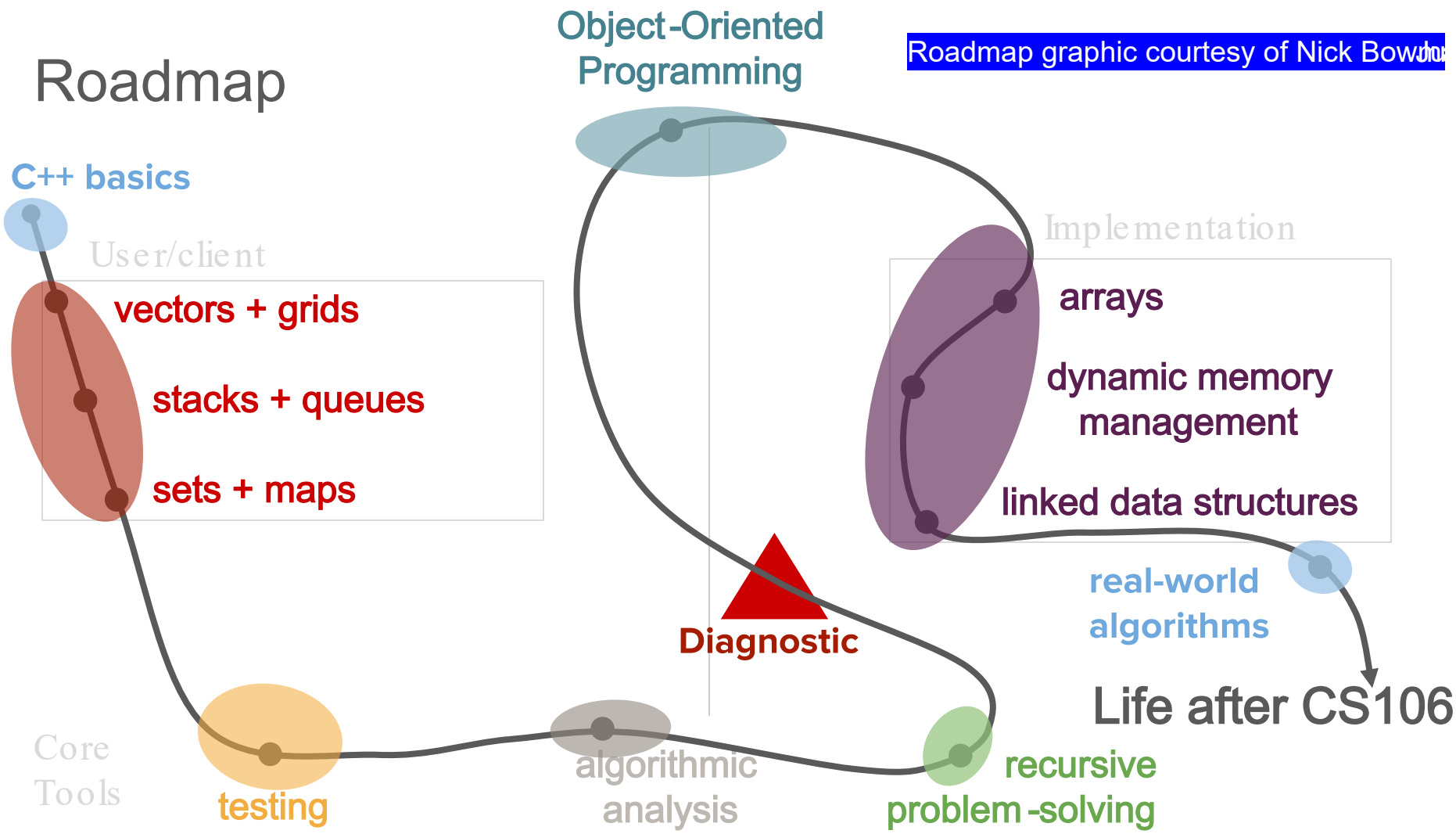
Recursive Backtracking: Enumeration

What is a game that would be easy to play if you had the ability to quickly think of all possible moves or plays?

(please put your answers in the chat)

Roadmap


Object-Oriented Programming
Roadmap graphic courtesy of Nick Bowdon
C++ basics
User/client
vectors + grids
stacks + queues
sets + maps
Implementation
arrays
dynamic memory management
linked data structures
real-world algorithms
Diagnostic
Life after CS106
Core Tools
testing
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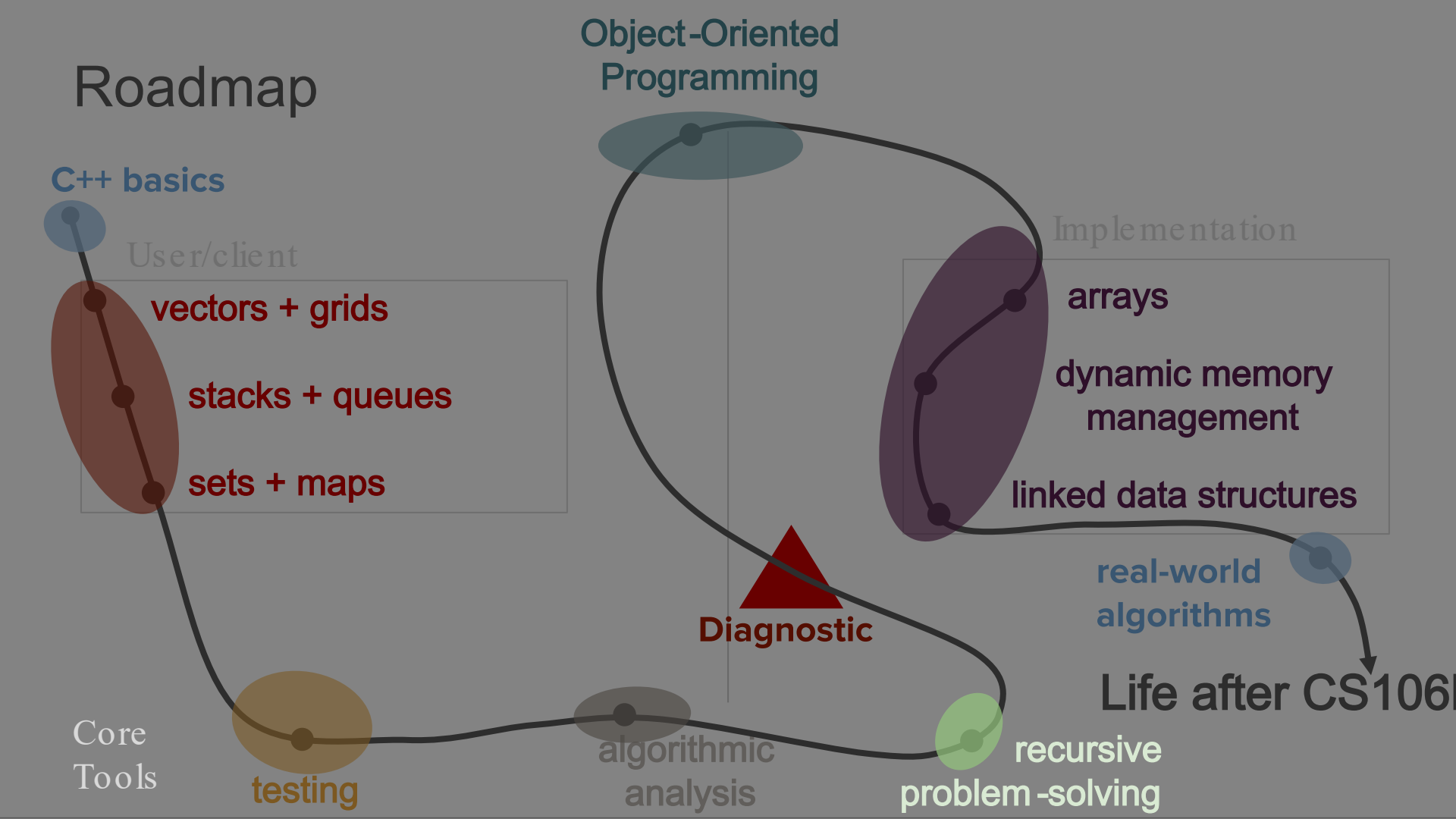
recursive problem-solving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Today's question

How can we leverage backtracking recursion to solve interesting problems?

Today's topics

1. Review
2. Shrinkable Words
3. Generating Subsets
4. Selecting Unbiased Juries

Celebrate Struggle

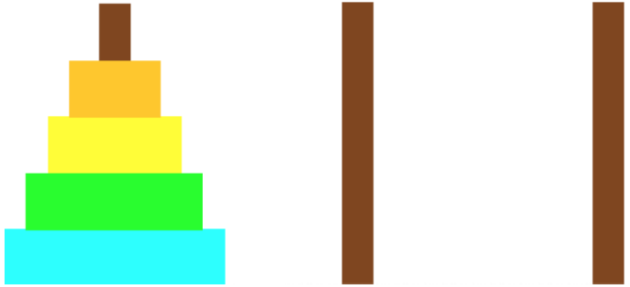
Review

(advanced recursion patterns and introduction to recursive backtracking)

Why do we use recursion?

- Elegance
  - Allows us to solve problems with very clean and concise code
- Efficiency
  - Allows us to accomplish better runtimes when solving problems
- Dynamic
  - Allows us to solve problems that are hard to solve iteratively

Elegance (Towers of Hanoi)



source auxiliary destination

```
void findSolution(int n, char source, char dest,
char aux) {
    if (n == 1) {
        moveSingleDisk(source, dest);
    } else {
        findSolution(n - 1, source, aux, dest);
        moveSingleDisk(source, dest);
        findSolution(n - 1, aux, dest, source);
    }
}
```

```
void findSolutionIterative(int n, char source, char dest, char aux) {
    int numMoves = pow(2, n) - 1; // total number of moves necessary

    // if number of disks is even, swap dest and aux posts
    if (n % 2 == 0) {
        char temp = dest;
        dest = aux;
        aux = temp;
    }

    Stack<int> srcStack;
    for (int i = n; i > 0; i--) {
        srcStack.push(i);
    }
    cout << srcStack << endl;
    Stack<int> destStack;
    Stack<int> auxStack;

    // Determine next move based on how many moves have been made so far
    for (int i = 1; i <= numMoves; i++) {
        switch (i % 3) {
            case 1:
                if (srcStack.isEmpty() ||
                        (!destStack.isEmpty() && srcStack.peek() > destStack.peek())) {
                    srcStack.push(destStack.pop());
                    moveSingleDisk(dest, source);
                } else {
                    destStack.push(srcStack.pop());
                    moveSingleDisk(source, dest);
                }
                break;
            case 2:
                if (srcStack.isEmpty() ||
                        (!auxStack.isEmpty() && srcStack.peek() > auxStack.peek())) {
                    srcStack.push(auxStack.pop());
                    moveSingleDisk(aux, source);
                } else {
                    auxStack.push(srcStack.pop());
                    moveSingleDisk(source, aux);
                }
                break;
            case 0:
                if (destStack.isEmpty() ||
                        (!auxStack.isEmpty() && destStack.peek() > auxStack.peek())) {
                    destStack.push(auxStack.pop());
                    moveSingleDisk(aux, dest);
                } else {
                    auxStack.push(destStack.pop());
                    moveSingleDisk(dest, aux);
                }
                break;
        }
    }
}
```

Efficiency (Binary Search)

- Leverage the structure in sorted data to **eliminate half of the search space every time** when searching for an element
  - Only do a direct comparison with the middle element in the list
  - Recursively search the left half if the element is less than the middle
  - Recursively search the right half if the element is greater than the middle

- Binary search has logarithmic Big-O: `O(log N)`
  - Enables efficient performance of sets and maps

Binary Search

Input Size	Runtime (s)
1000000	0.064
2000000	0.072
4000000	0.082
8000000	0.097
16000000	0.111
32000000	0.121
64000000	0.14

Linear Search

Input Size	Runtime (s)
10000	0.096
20000	0.189
40000	0.368
8000000	0.767
160000	1.387
320000	2.746
640000	6.154

Two types of recursion

Basic recursion

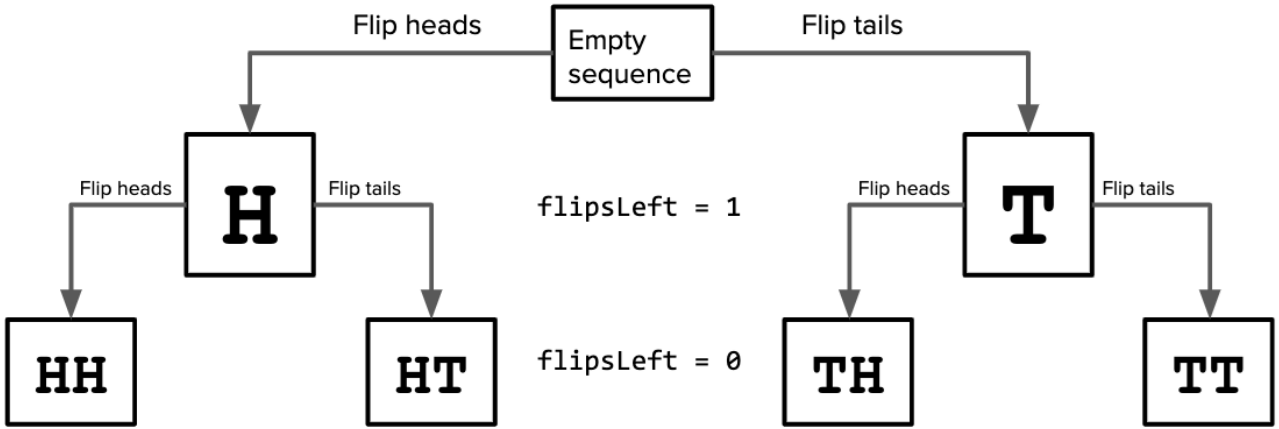
- One repeated task that builds up a solution as you come back up the call stack
- The final base case defines the initial seed of the solution and each call contributes a little bit to the solution
- Initial call to recursive function produces final solution

Backtracking recursion

- Build up many possible solutions through multiple recursive calls at each step
- Seed the initial recursive call with an “empty” solution
- At each base case, you have a potential solution

Dynamic (Coin Sequences + Decision Trees)

- The **height** of the tree corresponds to the **number of decisions** we have to make. The **width** at each decision point corresponds to the **number of options at each decision** .
- To exhaustively explore the entire search space, we must **try every possible option for every possible decision**.



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  - We can find one specific solution to a problem or prove that one exists
  - We can find the best possible solution to a given problem

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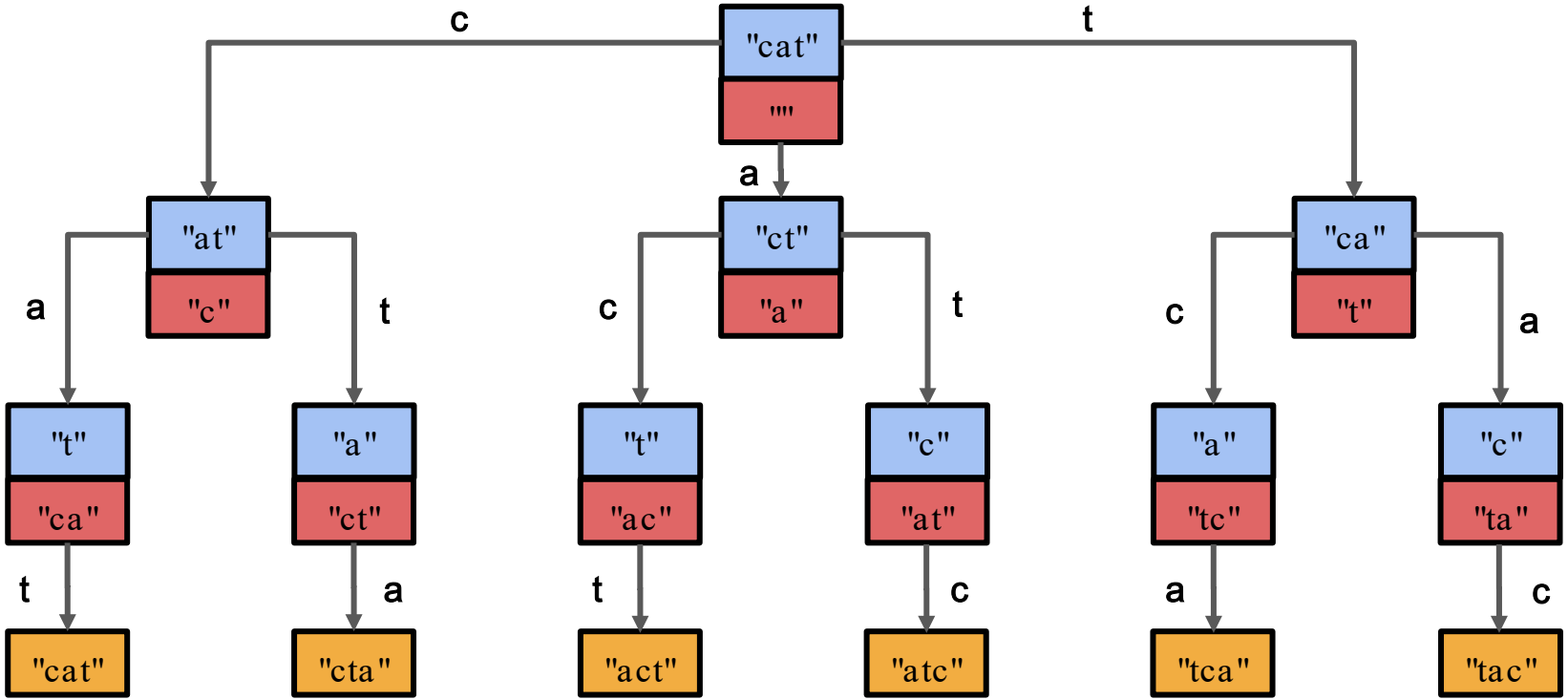
  - We can generate all possible solutions to a problem or count the total number of possible solutions to a problem
  - We can find one specific solution to a problem or prove that one exists
  - We can find the best possible solution to a given problem
- There are many, many examples of specific problems that we can solve, including
  - Generating permutations
  - Generating subsets
  - Generating combinations
  - And many, many more

Decisions yet to be made
Decisions made so far

Decision tree: Find all permutations of "cat"



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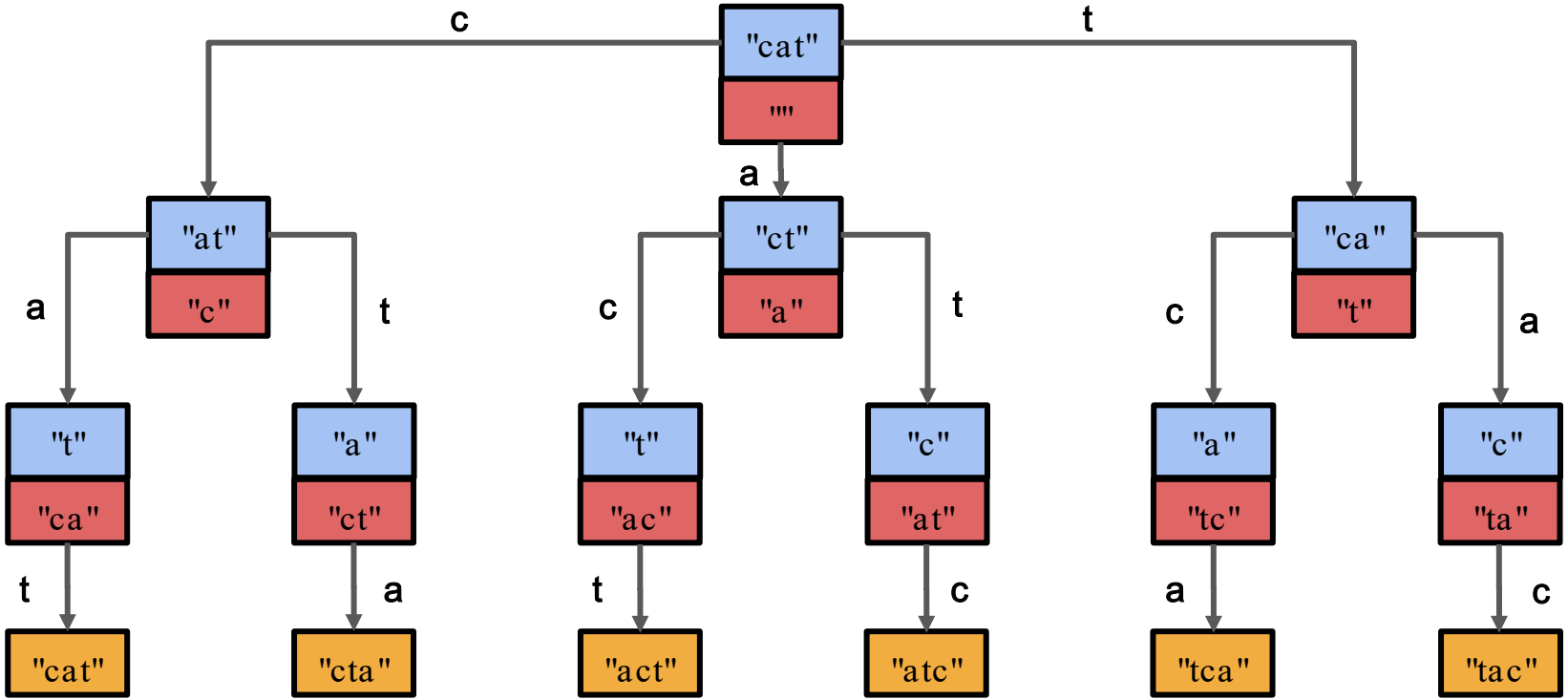
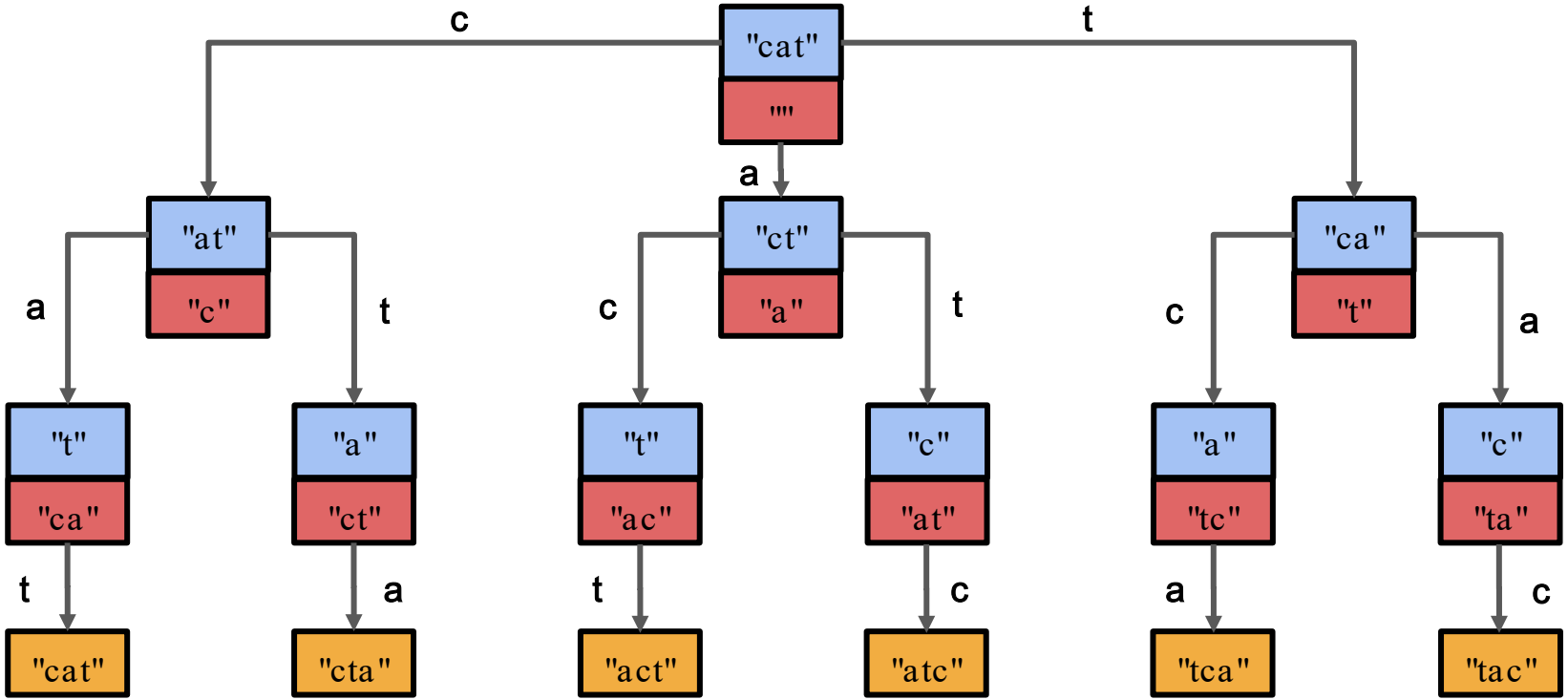


Base case: No letters remaining to choose!

Decisions yet to be made
Decisions made so far

Decision tree: Find all permutations of "cat"



Recursive case: For every letter remaining, add that letter to the current permutation and recurse!

Word scramble code

Permutations Code

```
void listPermutations(string s){
   listPermutationsHelper(s, "");
}

void listPermutationsHelper(string remaining, string soFar) {
   if (remaining.empty()) {
       cout << soFar << endl;
   } else {
       for (int i = 0; i < remaining.length(); i++) {
           char nextLetter = remaining[i];
           string rest = remaining.substr(0, i) + remaining.substr(i+1);
           listPermutationsHelper(rest, soFar + nextLetter);
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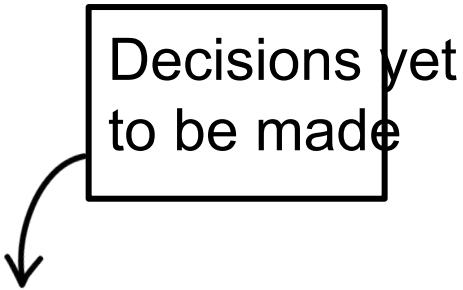
        }
    }
}
```

Use of recursive helper function with empty string as starting point

Permutations Code

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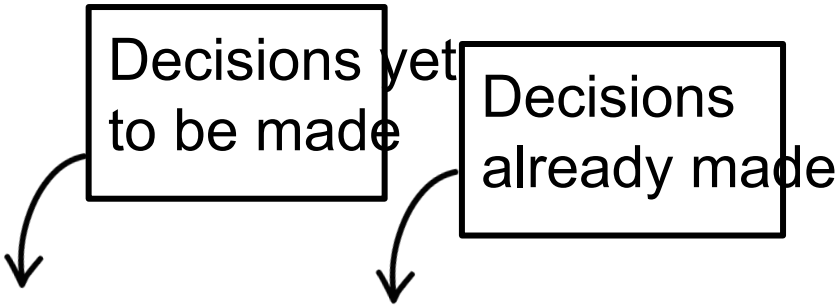


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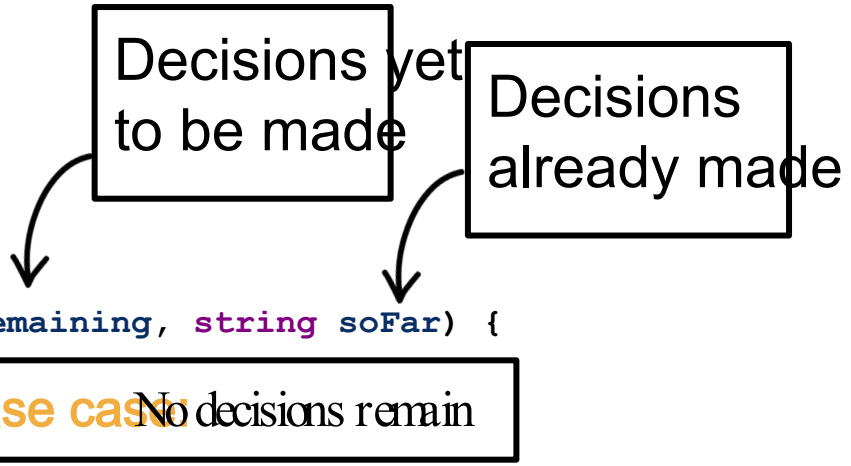

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        }
    }
}
```

Decisions yet to be made

Decisions already made

Base case: No decisions remain

Recursive case: Try all options for next decision

Takeaways

- The specific model of the general "choose / explore / unchoose" pattern in backtracking recursion that we applied here can be thought of as "copy, edit, recurse"
  - Since we passed all our parameters by value, each recursive stack frame had its own independent copy of the string data that it could edit as appropriate
  - The "unchoose" step is implicit since there is no need to undo anything by virtue of the fact that editing a copy only has local consequences.

Takeaways

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- Backtracking recursion can have variable branching factors at each level

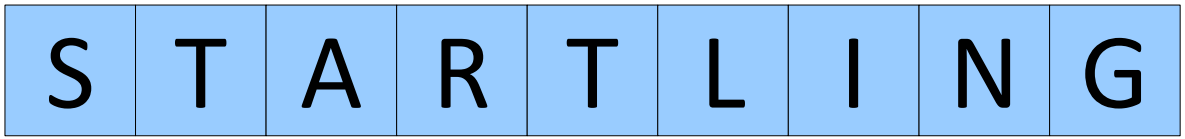
- Use of helper functions and initial empty params that get built up is common

How can we leverage backtracking recursion to solve interesting problems?

- A Little Word Puzzle

“What nine-letter word can be reduced to a single-letter word one letter at a time by removing letters, leaving it a legal word at each step?”

The Startling Truth?



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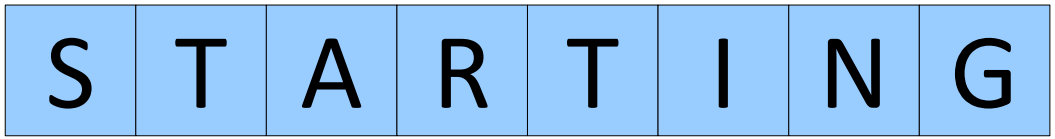

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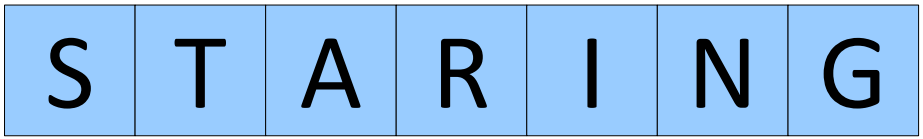

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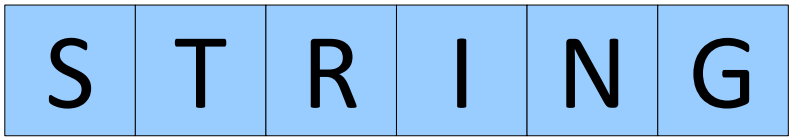

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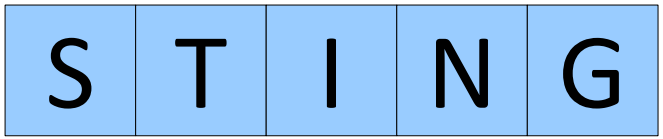

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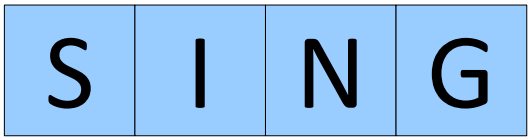

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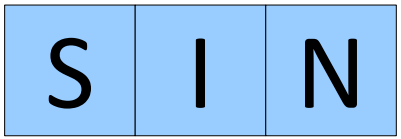

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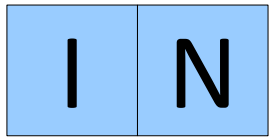

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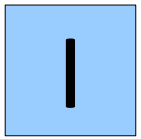

Is there *really* just **one**
nine-letter word with this property?

How can we determine if a word is shrinkable?

- A **shrinkable word** is a word that can be reduced down to one letter by removing one character at a time, leaving a word at each step.

- Idea: Let's use a decision tree to remove letters and determine **shrinkability** !

What defines our shrinkable decision tree?

- **Decision** at each step (each level of the tree):
  - What letter are going to remove?

- **Options** at each decision (branches from each node):
  - The remaining letters in the string

- Information we need to store along the way:
  - The shrinking string

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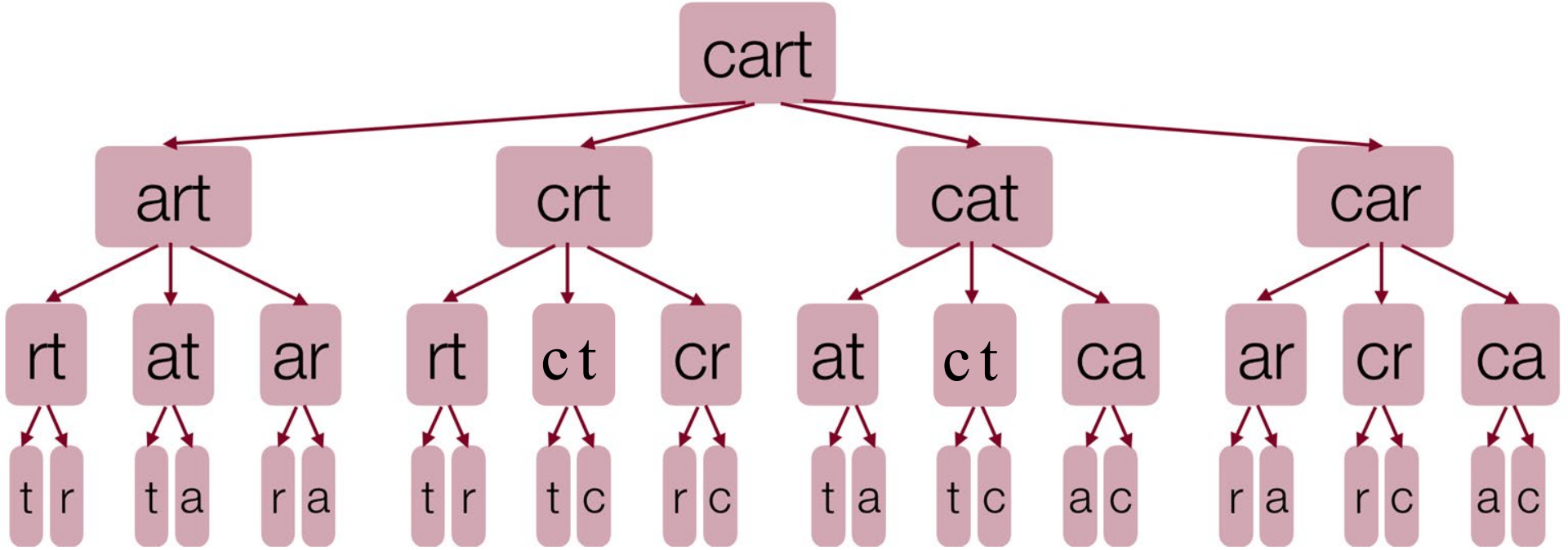

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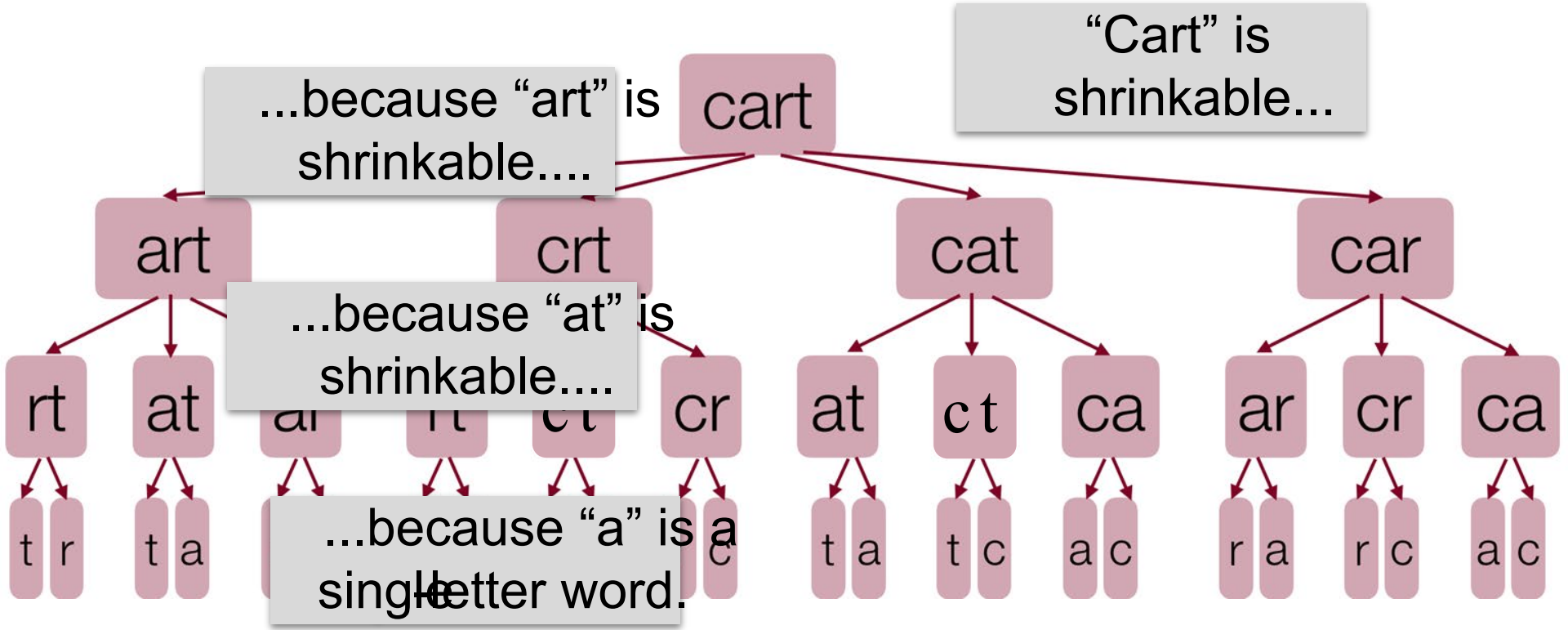

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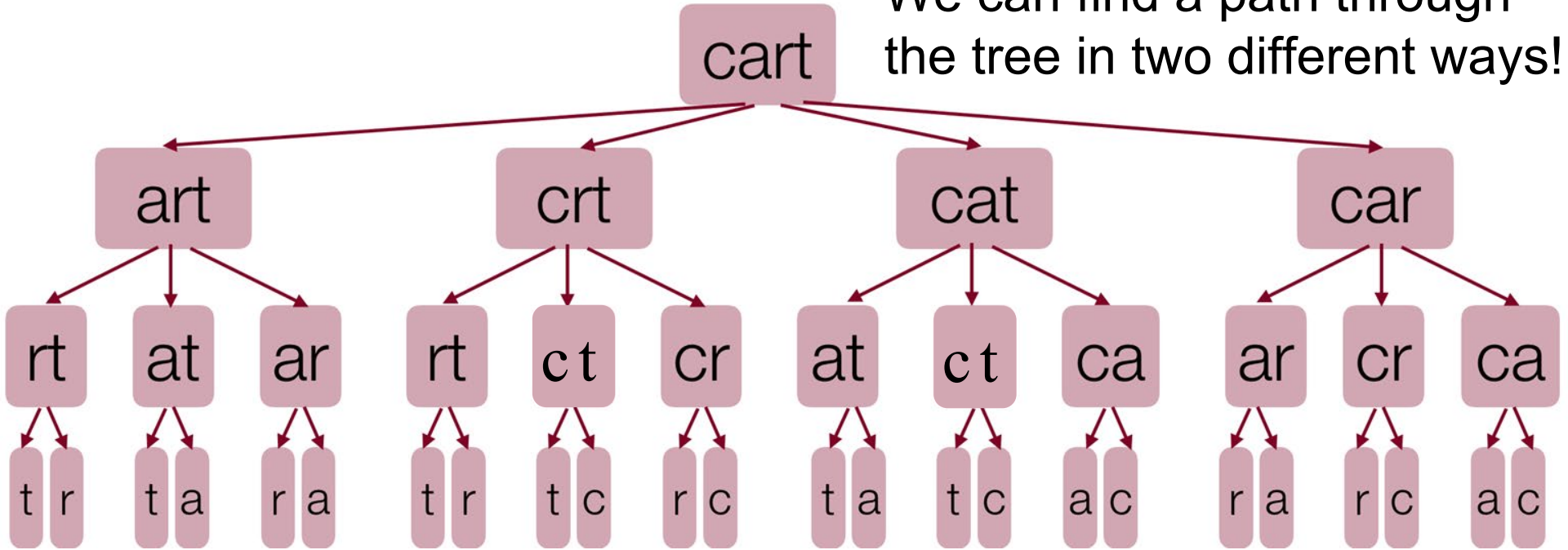

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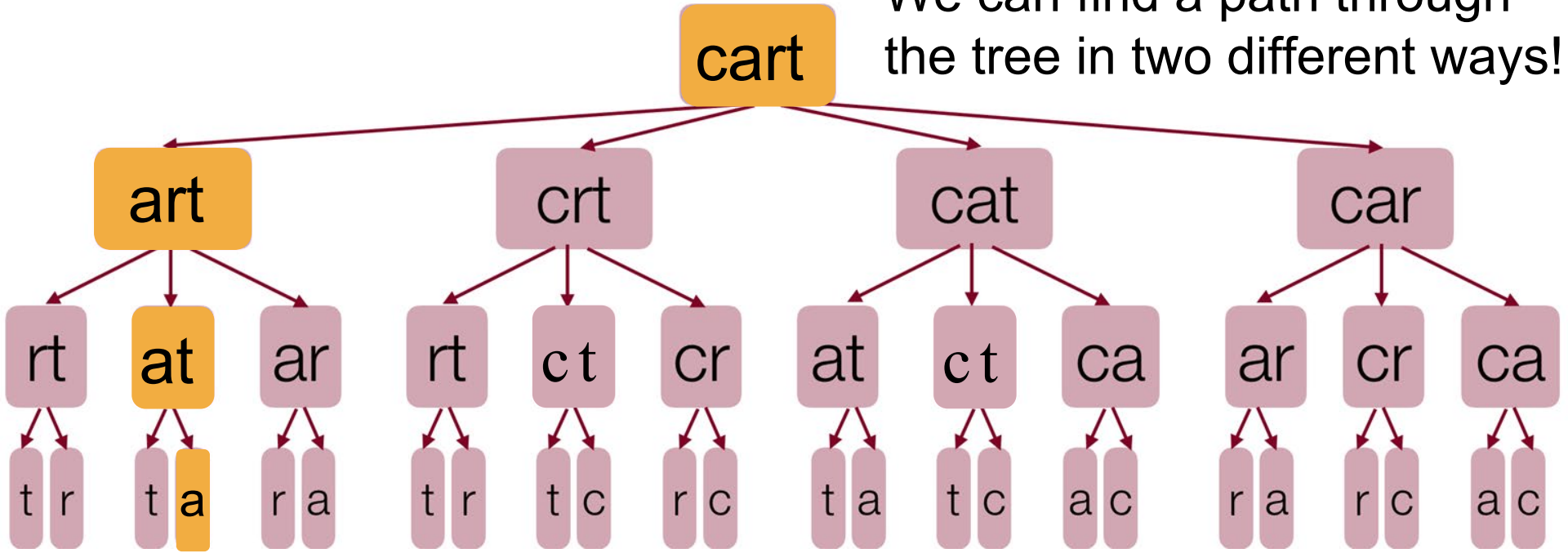

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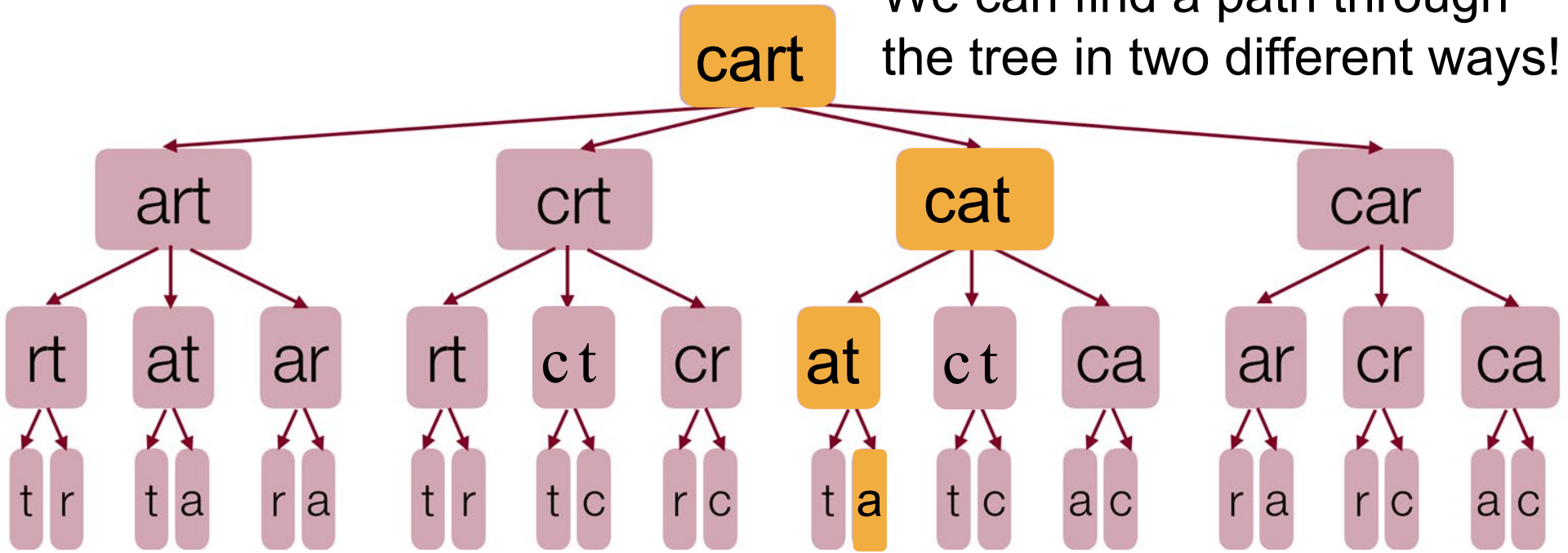
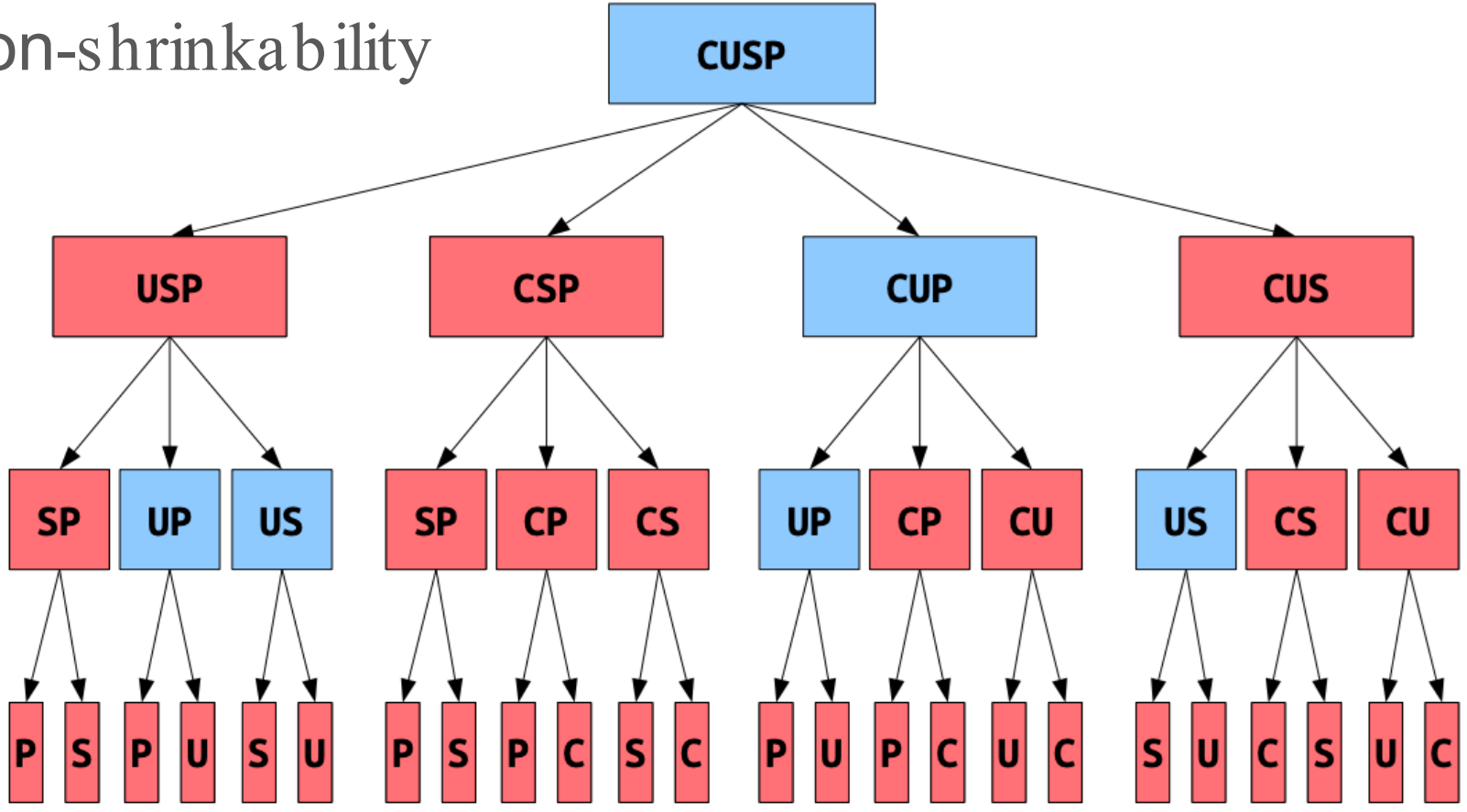


Non-shrinkability



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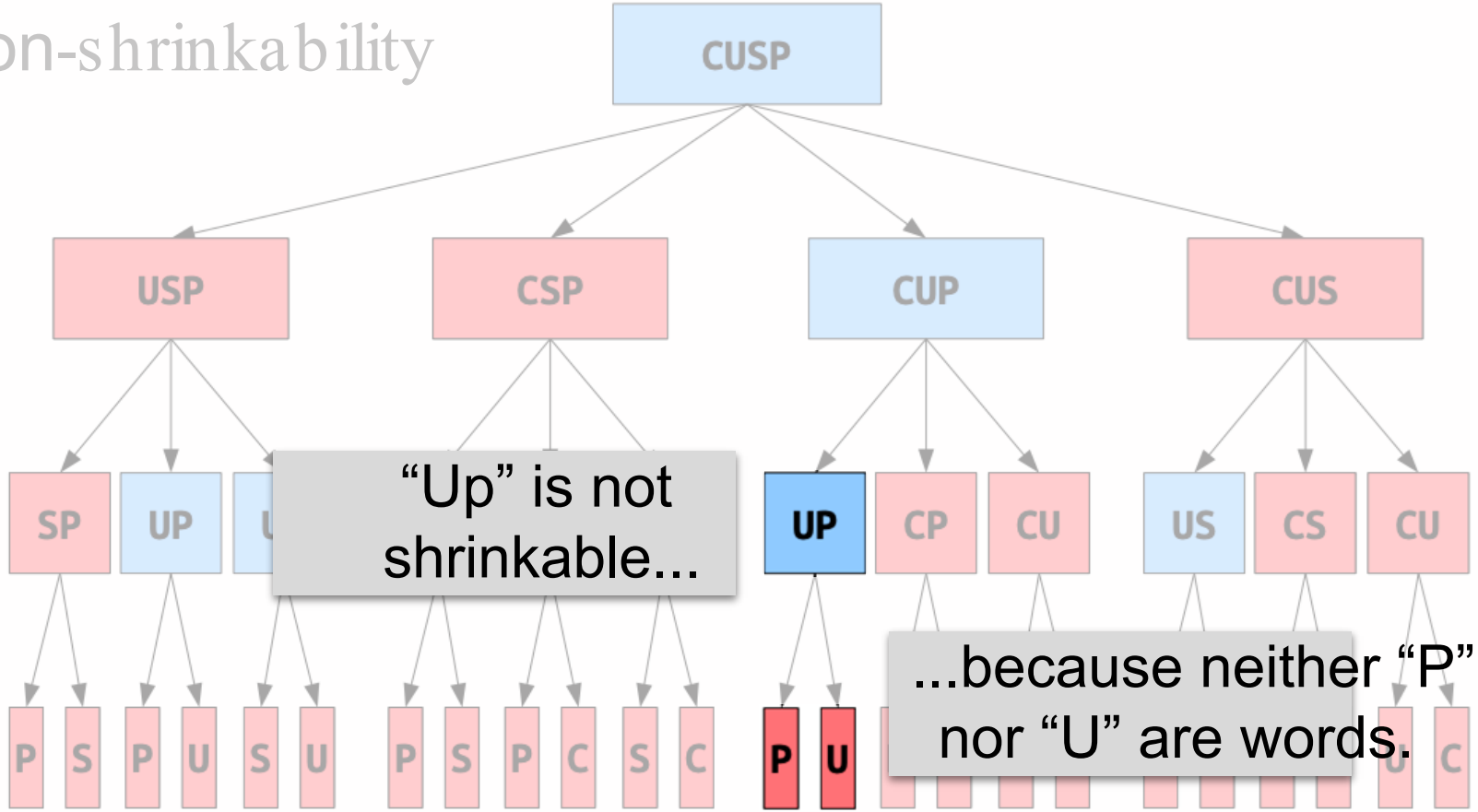

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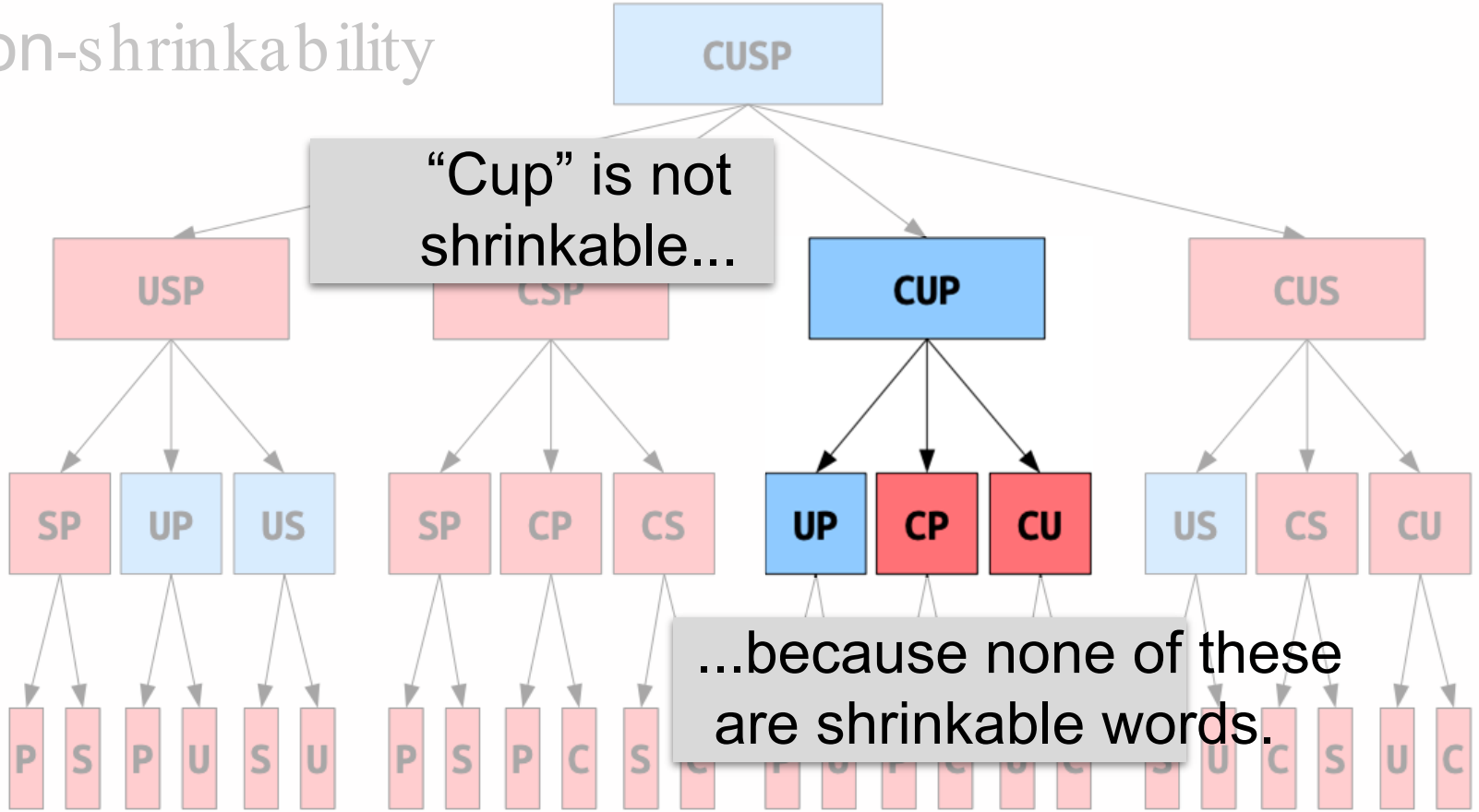
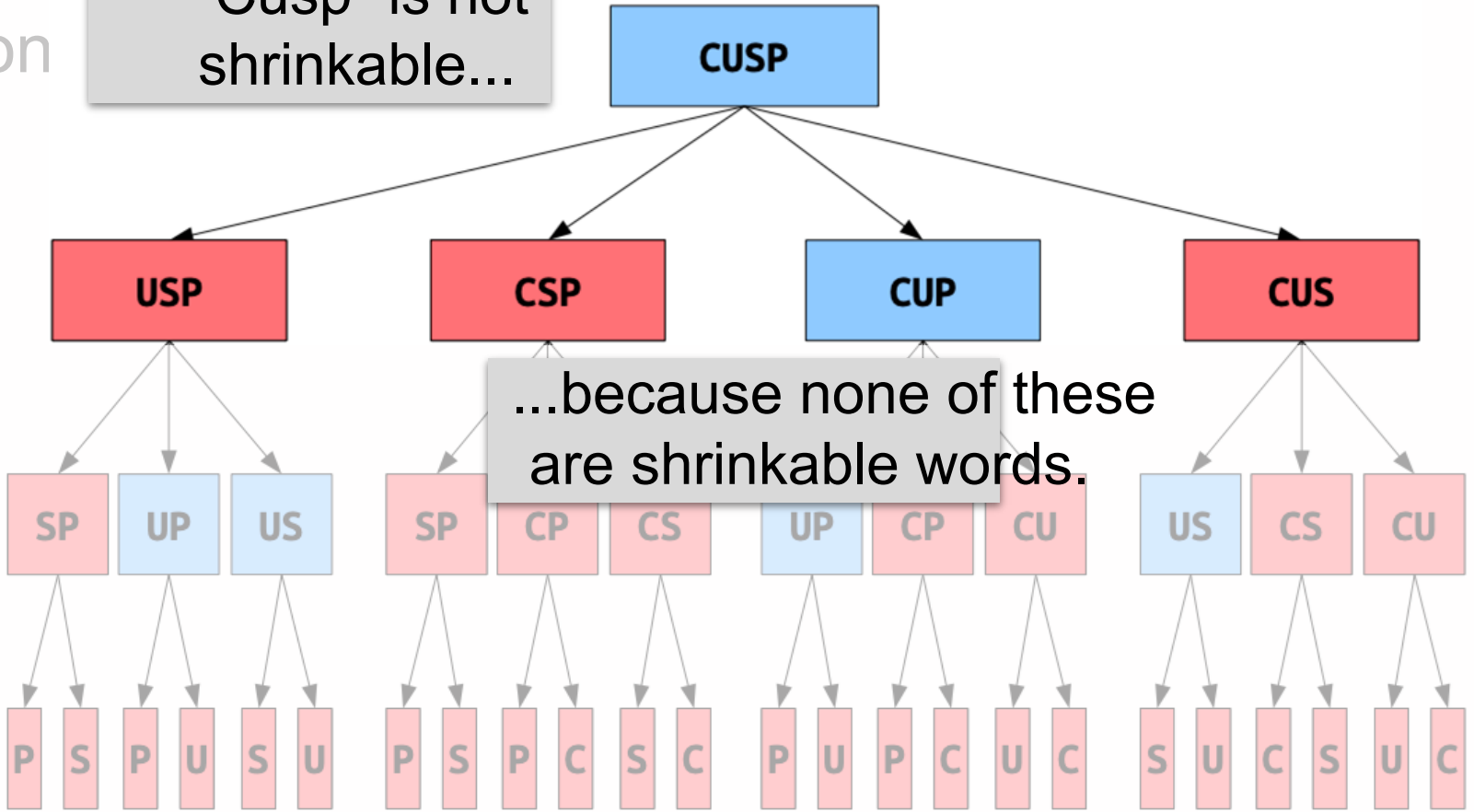


Non

“Cusp” is not shrinkable...



...because none of these are shrinkable words.

How can we determine if a word is shrinkable?

- **Base cases**:
  - A string that is not a word is not a shrinkable word.
  - Any single-letter word is shrinkable (A, I, and O).

- **Recursive cases**:
  - A multi-letter word is shrinkable if you can remove a letter to form a shrinkable word.
  - A multi-letter word is not shrinkable if no matter what letter you remove, it's not shrinkable.

Lexicon

- Lexicon is a helpful ADT provided by the Stanford C++ libraries (in **`lexicon.h`**) that is used specifically for storing many words that make up a dictionary

- Generally, Lexicons offer faster lookup than normal Sets, which is why we choose to use them when dealing with words and large dictionaries

- ```
  Lexicon lex("res/EnglishWords.txt"); // create from file
  lex.contains("koala"); // returns true
  lex.contains("zzzzz"); // returns false
  lex.containsPrefix("fi"); // returns true if there are any words starting with "fi" in the dictionary
  ```
  ```

Let’s code it!

Takeaways

- This is another example of **copy-edit-recurse** to choose, explore, and then implicitly unchoose!

- In this problem, we're using backtracking to **find if a solution exists** .
  - Notice the way the recursive case is structured:

```
for all options at each decision point:
        if recursive call returns true:
                return true;
return false if all options are exhausted;
```

Announcements

Announcements

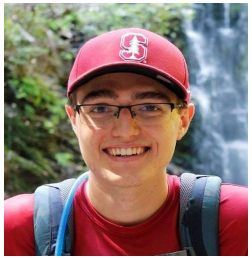
- Assignment 3 was released last Thursday.
  - The Assignment 3 YEAH session slides and recording have been posted.
  - This assignment is challenging and quite long; please come to LaIR or office hours to check in with the teaching team if you need anything!

- We've released practice problems and information about the diagnostic. You'll be able to take the diagnostic over the weekend.
  - Please make sure to read the instructions on the diagnostic page of the website, and verify that you can access your Gradescope account before the weekend!

Subsets

Subsets

Given a group of people, suppose we wanted to generate all possible teams, or subsets, of those people:



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Even though we may no
care about this “team,” th
empty set is a subset of o
original set!

Subsets

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As humans, it might be easiest to think about all teams (subsets) of a particular size.

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{“Nick”}
{“Kylie”}
{“Trip”}
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{“Nick”, “Kylie”}
{“Nick”, “Trip”}
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{“Nick”, “Kylie”}
{“Nick”, “Trip”}
{“Kylie”, “Trip”}
{“Nick”, “Kylie”, “Trip”}
```

Another case of “generate/count all solutions” using recurs backtracking!

Subsets

Given a group of people, suppose we wanted to generate all possible teams, or subsets, of those people:

```
{}
{“Nick”}
{“Kylie”}
{“Trip”}
{“Nick”, “Kylie”}
{“Nick”, “Trip”}
{“Kylie”, “Trip”}
{“Nick”, “Kylie”, “Trip”}
```

For computers generati
subsets (and thinking ab
decisions), there’s anoth
pattern we might notice

Subsets

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{“Nick”}
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{“Nick”, “Kylie”}
{“Nick”, “Trip”}
{“Kylie”, “Trip”}
{“Nick”, “Kylie”, “Trip”}
```

Half the subsets contain
“Nick”

Subsets

Given a group of people, suppose we wanted to generate all possible teams, or subsets, of those people:

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{“Nick”}
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{“Trip”}
{“Nick”, “Kylie”}
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{“Nick”, “Kylie”, “Trip”}
```

Half the subsets contain “Kylie”

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{“Kylie”, “Trip”}
{“Nick”, “Kylie”, “Trip”}
```

Half the subsets contain
“Trip”

Subsets

Given a group of people, suppose we wanted to generate all possible teams, or subsets, of those people:

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{“Trip”}
{“Nick”, “Kylie”}
{“Nick”, “Trip”}
{“Kylie”, “Trip”}
{“Nick”, “Kylie”, “Trip”}
```

Half the subsets that contain “Trip” also conta
“Nick”

Subsets

Given a group of people, suppose we wanted to generate all possible teams, or subsets, of those people:

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{“Nick”}
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{“Nick”, “Trip”}
{“Kylie”, “Trip”}
{“Nick”, “Kylie”, “Trip”}
```

Half the subsets that contain both “Trip” and “Nick” contain “Kylie”

Subsets

Given a group of people, suppose we wanted to generate all possible teams, or subsets, of those people:

```
{}
{“Nick”}
{“Kylie”}
{“Trip”}
{“Nick”, “Kylie”}
{“Nick”, “Trip”}
{“Kylie”, “Trip”}
{“Nick”, “Kylie”, “Trip”}
```

It’s time to draw a tree!!!

What defines our subsets decision tree?

- **Decision** at each step (each level of the tree):
  - Are we going to include a given element in our subset?

What defines our subsets*decision tree*?

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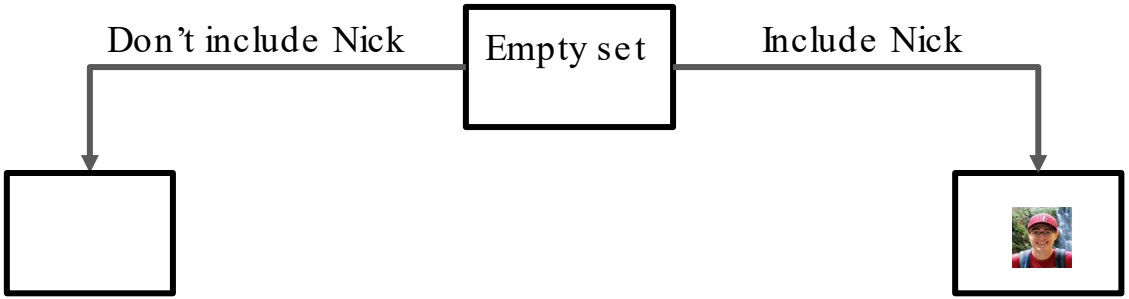
Decision tree



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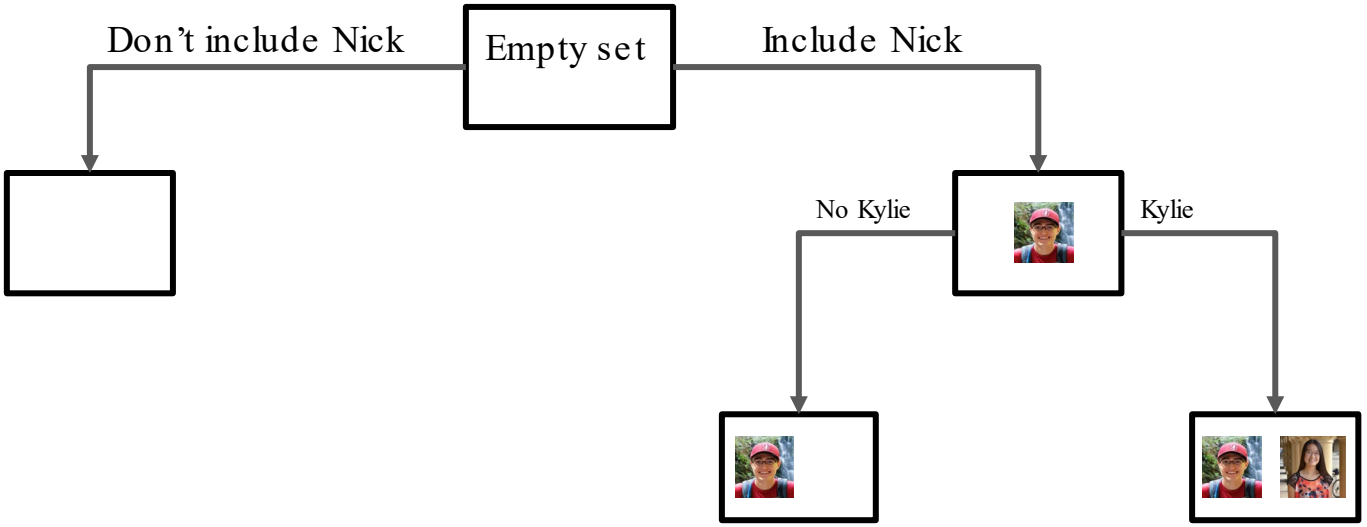

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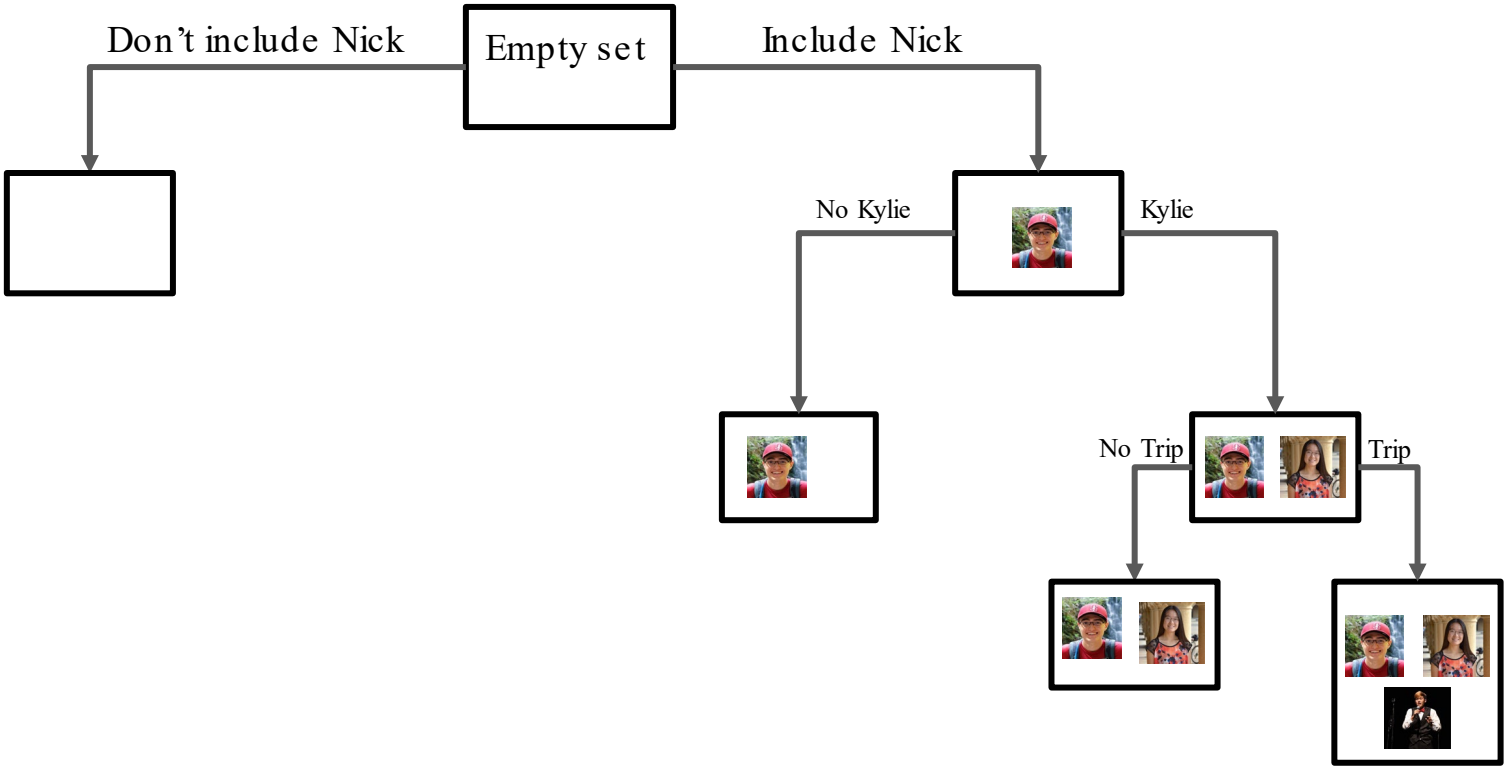

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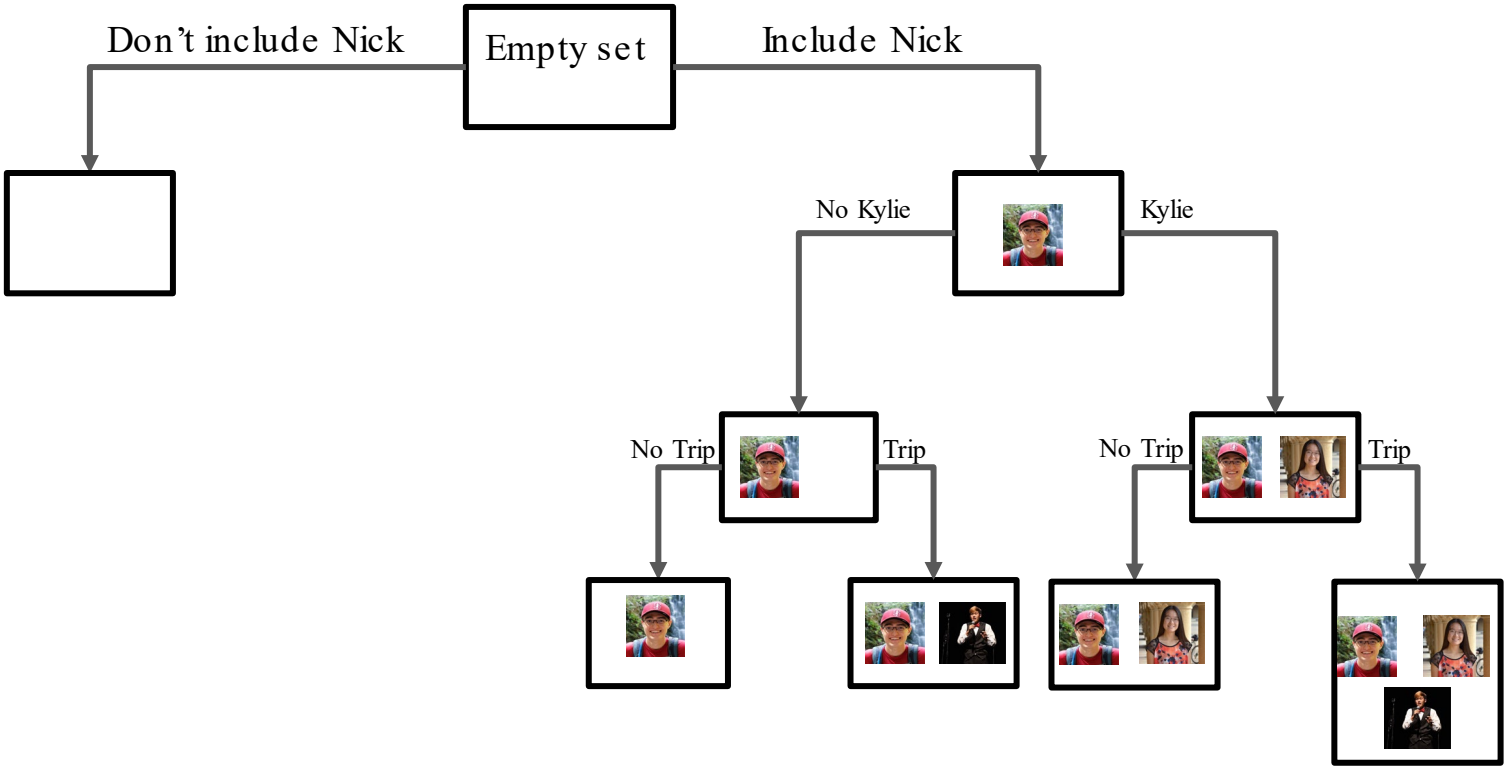
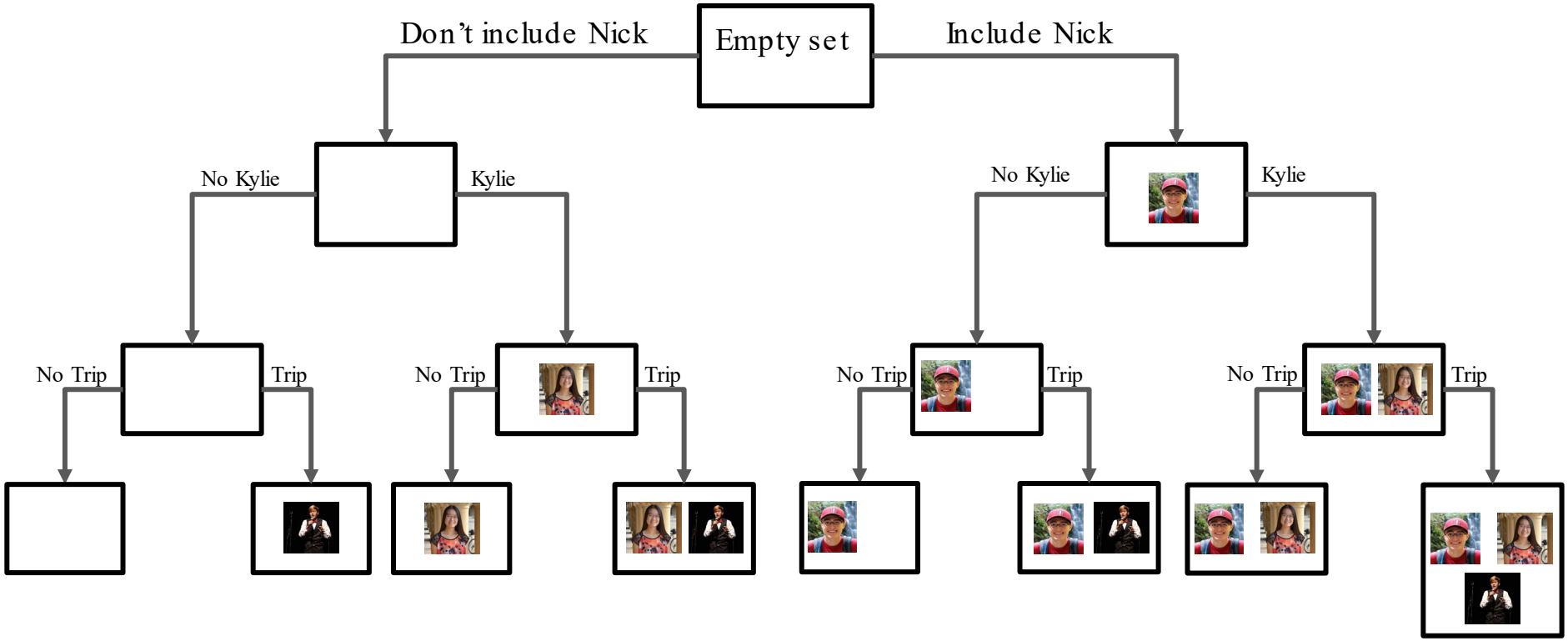


Decision tree



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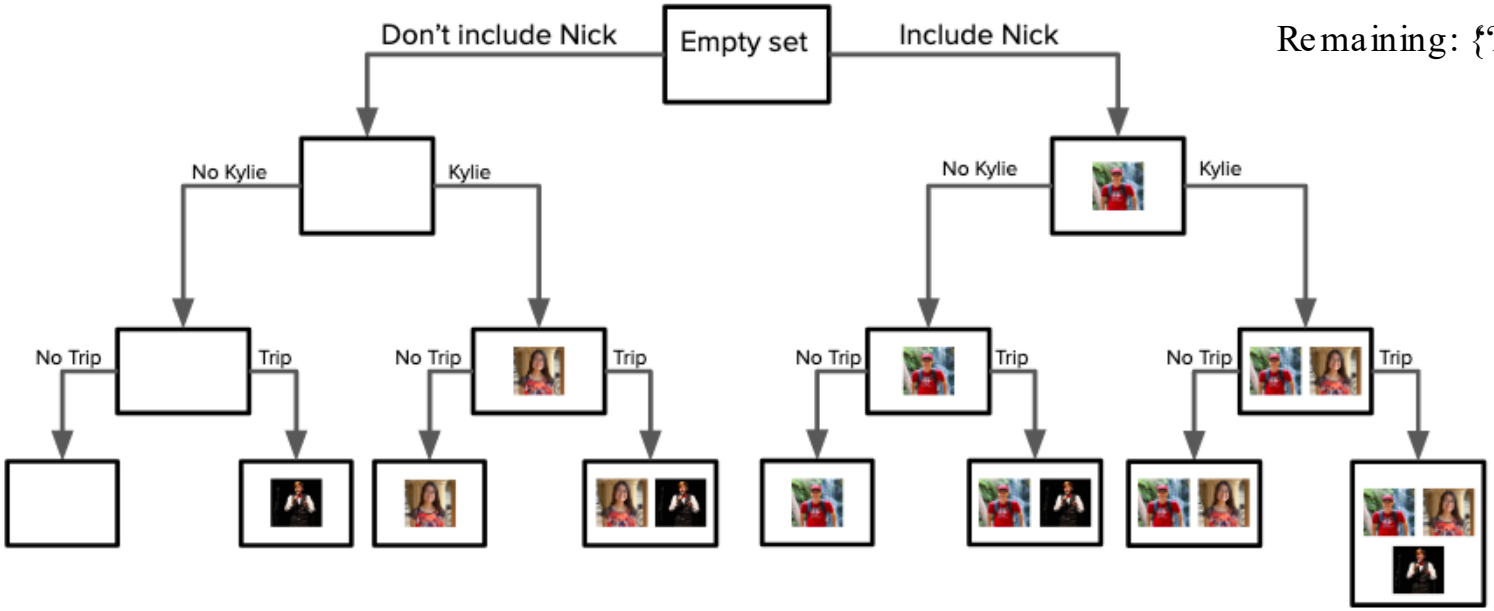
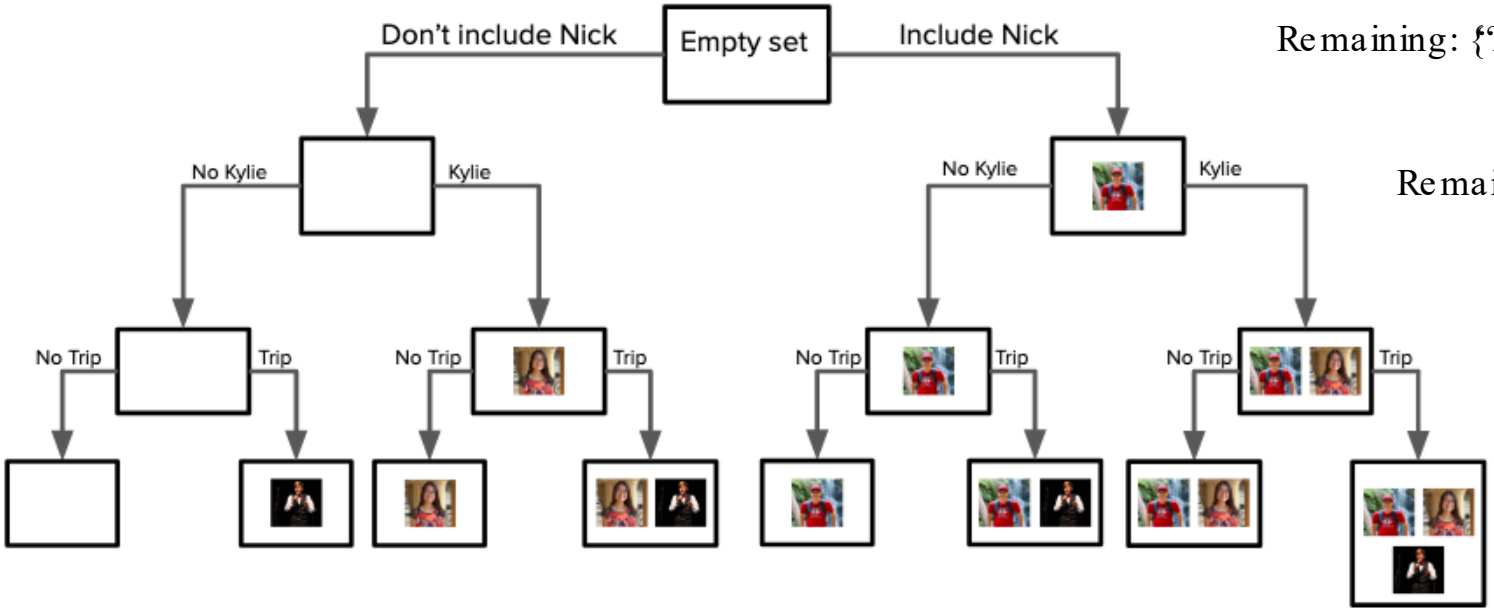


Remaining: {"Nick", "Kylie", "Trip"}

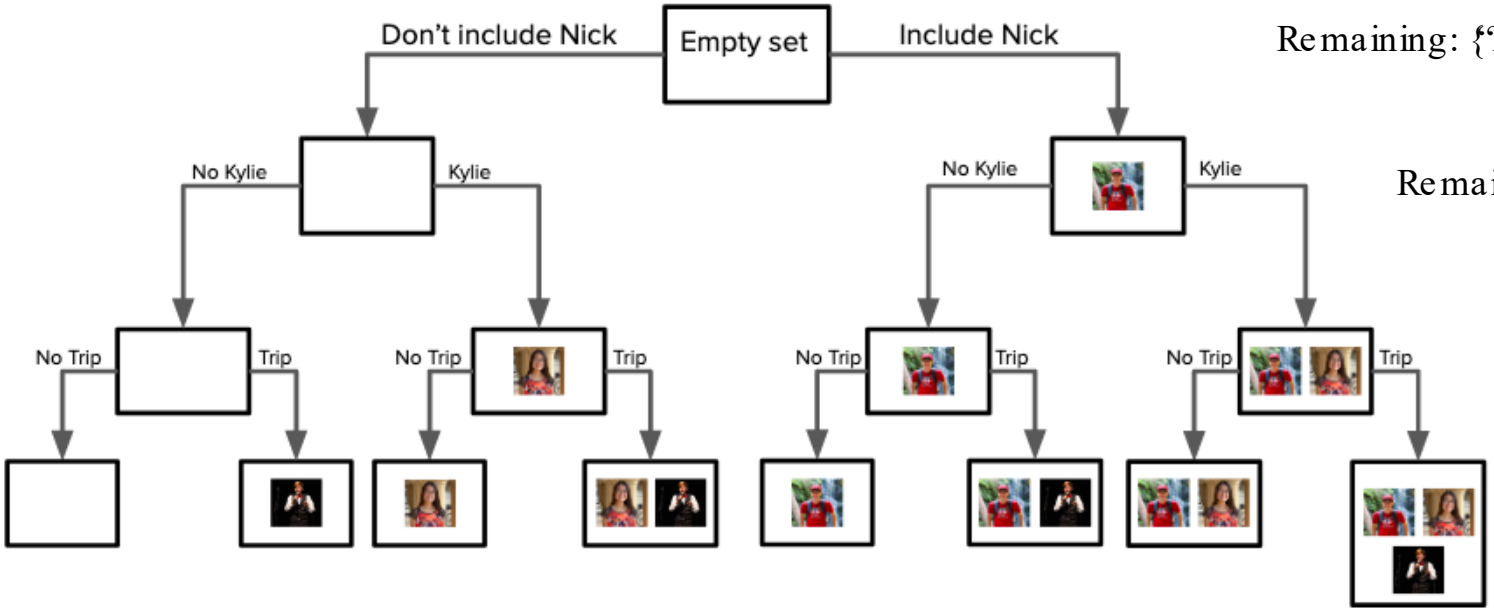
Decision tree



Remaining: {"Nick", "Kylie", "Trip"}

Remaining: {"Kylie", "Trip"}

Decision tree



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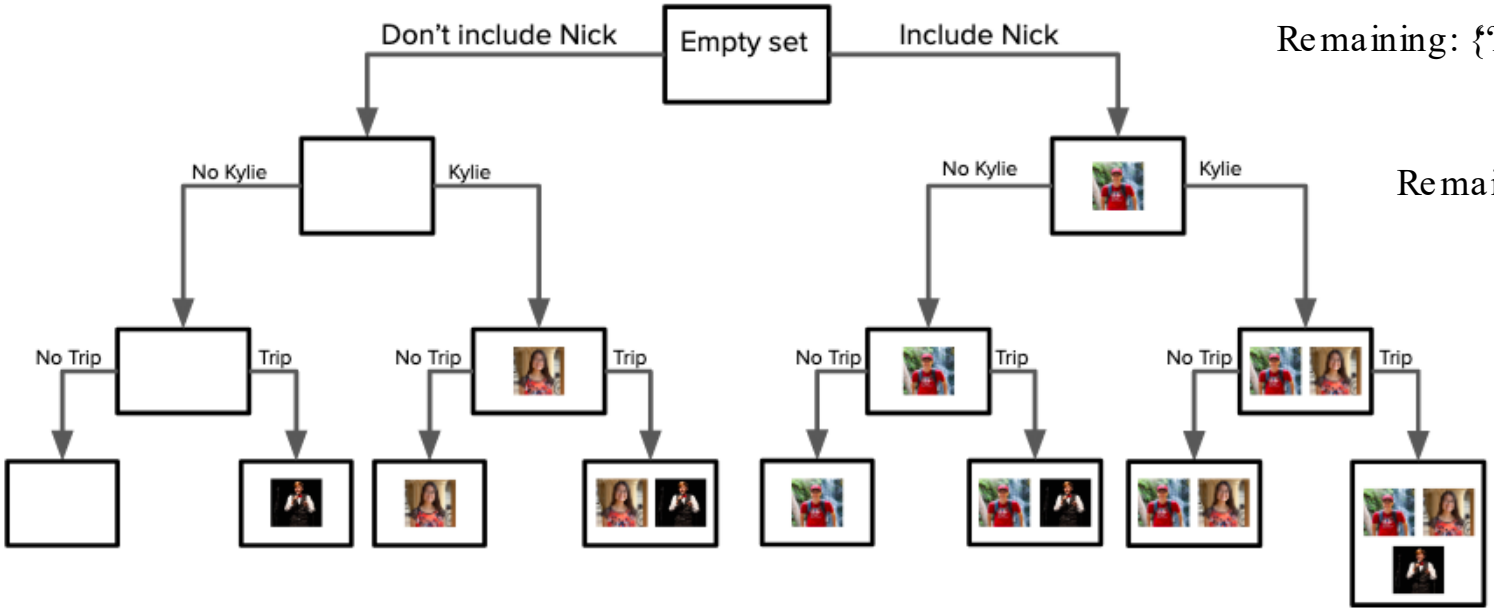

Remaining: {"Nick", "Kylie", "Trip"}

Remaining: {"Kylie", "Trip"}

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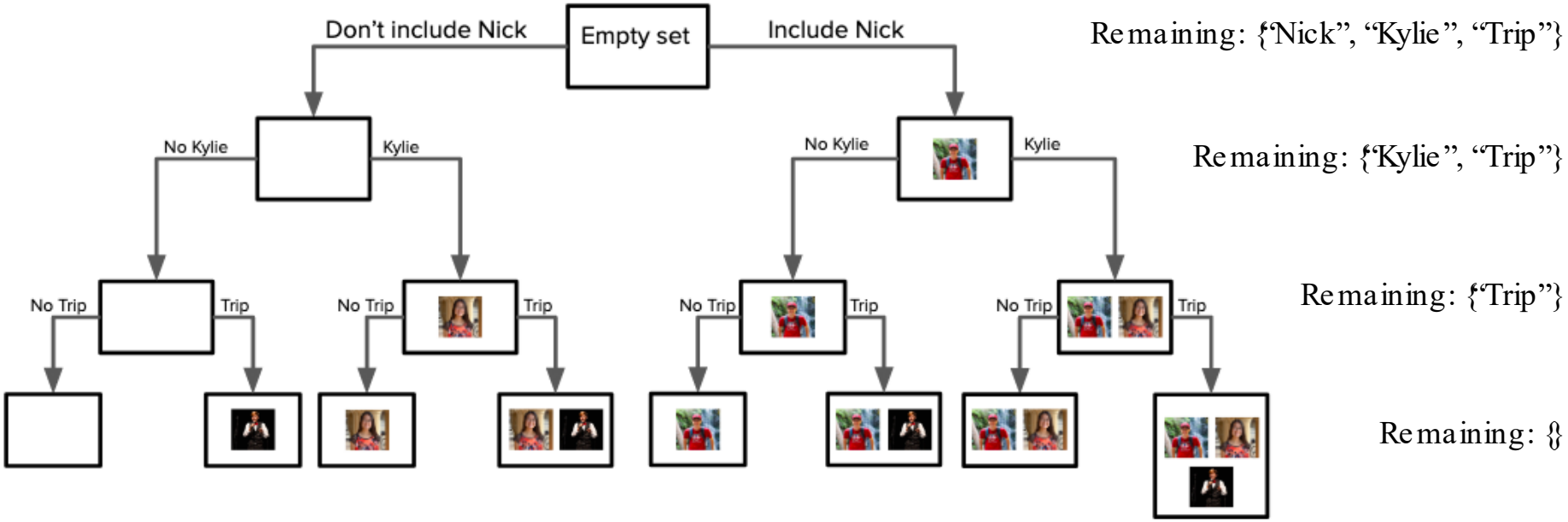
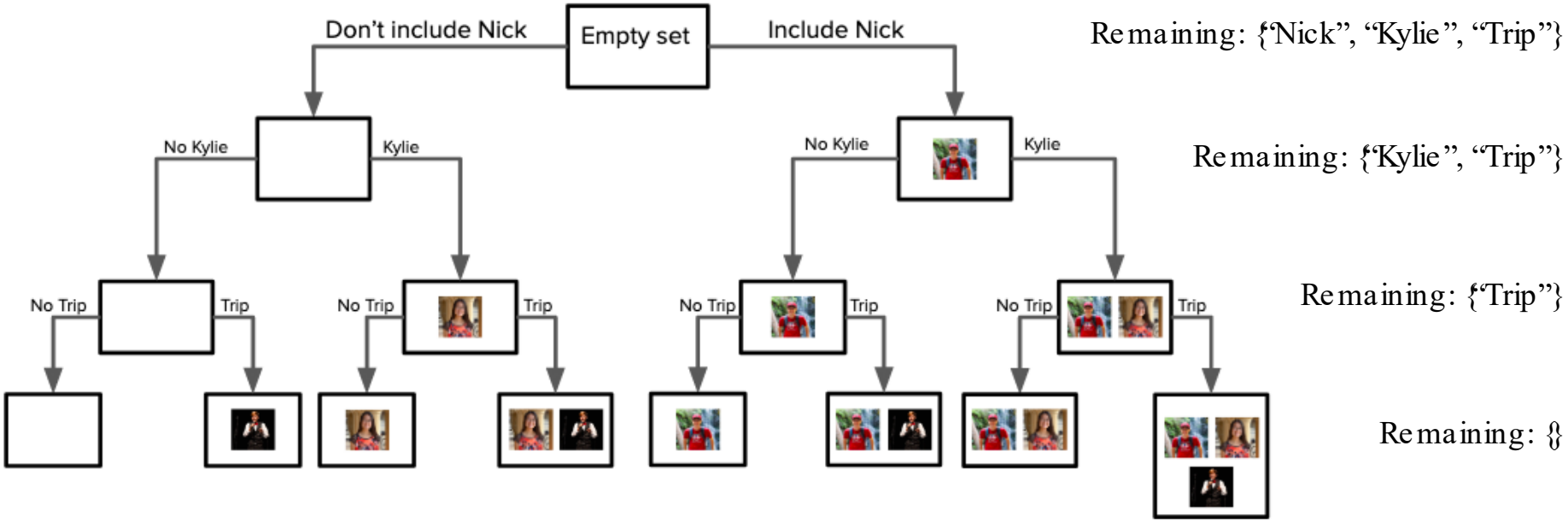
Decision tree



Base case: No people remaining to choose from!

Decision tree



Recursive case: Pick someone in the set. Choose to include or not include them.

How do recursive backtracking solutions look different when data structures are involved?

Let’s code it!

Takeaways

- This is our first time seeing an explicit “unchoose” step
  - This is necessary because we’re passing sets by reference and editing them!

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```
string elem = remaining.first();
// remove this element from possible choices
remaining = remaining - elem;
listSubsetsHelper(remaining, chosen); // do not add elem to chosen
chosen = chosen + elem;
listSubsetsHelper(remaining, chosen); // add elem to chosen
chosen = chosen - elem;
// add this element back to possible choices
remaining = remaining + elem;
```

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Choose

```
string elem = remaining.first();
// remove this element from possible choices
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listSubsetsHelper(remaining, chosen); // do not add elem to chosen
chosen = chosen + elem;
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// add this element back to possible choices
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```

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Explore (part 1)

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Explore (part 2)

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listSubsetsHelper(remaining, chosen); // add elem to chosen
chosen = chosen - elem;
// add this element back to possible choices
remaining = remaining + elem;
```

Takeaways

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Explicit Unchoose (i.e. undo)

```
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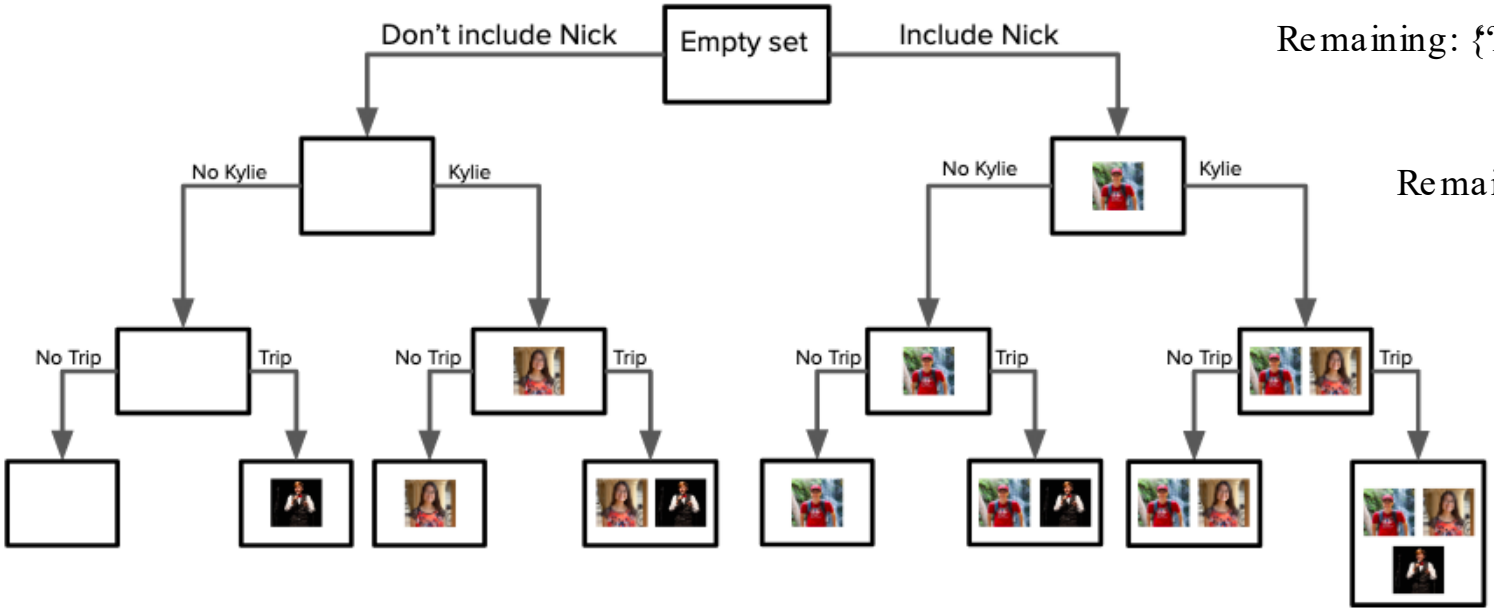
listSubsetsHelper(remaining, chosen); // add elem to chosen
chosen = chosen - elem;
// add this element back to possible choices
remaining = remaining + elem;
```

Decision tree



Remaining: {"Nick", "Kylie", "Trip"}

Remaining: {"Kylie", "Trip"}

Remaining: {"Trip"}

Remaining: {}

Takeaways

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Without this step, we could not explore the other side of the tree

```
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listSubsetsHelper(remaining, chosen); // add elem to chosen
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```

Takeaways

- This is our first time seeing an explicit “unchoose” step
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- It’s important to consider not only decisions and options at each decision, but also to keep in mind what information you have to keep track of with each recursive call. This might help you define your base case.

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- The subset problem contains themes we’ve seen in backtracking recursion:
  - Building up solutions as we go down the decision tree
  - Using a helper function to abstract away implementation details

Application: Choosing an Unbiased Jury

Jury Selection

- The process of jury selection involves processing a large pool of candidates that have been called for jury duty, and selecting some small subset of those candidates to serve on the jury.


12 ANGRY MEN

Photo from 2007 TV show, the *Verdict*

Jury Selection

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- When selecting members of a jury, each individual person will come in with their own biases that might sway the case.
- Ideally, we would like to select a jury that is unbiased (sum of all biases is 0)
- An unbiased jury is just a subset with a special property – let's apply the code that we just wrote!

What defines our subsets decision tree?

- **Decision** at each step (each level of the tree):
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What defines our jury selection decision tree?

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  - Are we going to include a given candidate in our jury?

- **Options** at each decision (branches from each node):
  - Include candidate
  - Don't include candidate

- Information we need to store along the way:
  - The collection of candidates making up our jury so far
  - The remaining candidates to consider
  - The sum total bias of the current jury so far

Jury Selection Pseudocode

- Problem Setup
  - Assume that we have defined a custom **`juror`** struct, which packages up important information about a juror (their **`name`** and their **`bias`**, represented as an **`int`**)

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- Recursive Case
  - Select a candidate that hasn't been considered yet.
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  - Select a candidate that hasn't been considered yet.
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  - Try including them in the jury, and recursively find all possible unbiased juries.

- Base Case
  - Once we're out of candidates to consider, check the bias of the current jury. If 0, display them!

Jury Selection Code

```
void findAllUnbiasedJuriesHelper(Vector<juror>& allCandidates, Vector<juror>& currentJury, int
currentBias){
    if (allCandidates.isEmpty()){
        if (currentBias == 0){
            displayJury(currentJury);
        }
    } else {
        juror currentCandidate = allCandidates[0];
        allCandidates.remove(0);

        findAllUnbiasedJuriesHelper(allCandidates, currentJury, currentBias);
        currentJury.add(currentCandidate);
        findAllUnbiasedJuriesHelper(allCandidates, currentJury, currentBias + currentCandidate.bias);
        currentJury.remove(currentJury.size() - 1);

        allCandidates.insert(0, currentCandidate);
    }
}

void findAllUnbiasedJuries(Vector<juror>& allCandidates){
    Vector<juror> jury;
    findAllUnbiasedJuriesHelper(allCandidates, jury, 0);
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Helper function: Extra variable to keep track of total bias

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Base case: Only display juries with no total bias

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```

Recursive case: Consider juries both with and without this person

Jury Selection
Optimization

Jury Selection Code

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}
```

Vector addition/removal can be an expensive operation. Can we do better?

Optimizing Subset Creation

- The core component of subset generation includes visiting each element once, and making a decision about whether to include it or not

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- Previously, we have done so by arbitrarily picking the "first" element in the collection as the one under consideration, and then removed it (expensive) from the collection for future recursive calls.

Optimizing Subset Creation

- The core component of subset generation includes visiting each element once, and making a decision about whether to include it or not

- Previously, we have done so by arbitrarily picking the "first" element in the collection as the one under consideration, and then removed it (expensive) from the collection for future recursive calls.

- **Key Idea:** Instead of modifying the collection of elements, let's just keep track of our current place in the collection (**index of the element that is currently under consideration**).

Jury Selection Code v2.0

```
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        }
    } else {
        juror currentCandidate = allCandidates[0];
        allCandidates.remove(0);

        findAllUnbiasedJuriesHelper(allCandidates, currentJury, currentBias);
        currentJury.add(currentCandidate);
        findAllUnbiasedJuriesHelper(allCandidates, currentJury, currentBias + currentCandidate.bias);
        currentJury.remove(currentJury.size() - 1);

        allCandidates.insert(0, currentCandidate);
    }
}

void findAllUnbiasedJuries(Vector<juror>& allCandidates){
    Vector<juror> jury;
    findAllUnbiasedJuriesHelper(allCandidates, jury, 0);
}
```

Jury Selection Code v2.0

```
void findAllUnbiasedJuriesHelper(Vector<juror>& allCandidates, Vector<juror>& currentJury,
                                 int currentBias, int index){
    if (allCandidates.isEmpty()){
        if (currentBias == 0){
            displayJury(currentJury);
        }
    } else {
        juror currentCandidate = allCandidates[0];
        allCandidates.remove(0);

        findAllUnbiasedJuriesHelper(allCandidates, currentJury, currentBias);
        currentJury.add(currentCandidate);
        findAllUnbiasedJuriesHelper(allCandidates, currentJury, currentBias + currentCandidate.bias);
        currentJury.remove(currentJury.size() - 1);

        allCandidates.insert(0, currentCandidate);
    }
}

void findAllUnbiasedJuries(Vector<juror>& allCandidates){
    Vector<juror> jury;
    findAllUnbiasedJuriesHelper(allCandidates, jury, 0, 0);
}
```

Jury Selection Code v2.0

```
void findAllUnbiasedJuriesHelper(Vector<juror>& allCandidates, Vector<juror>& currentJury, int
currentBias, int index){
    if (index == allCandidates.size()){
        if (currentBias == 0){
            displayJury(currentJury);
        }
    } else {
        juror currentCandidate = allCandidates[0];
        allCandidates.remove(0);

        findAllUnbiasedJuriesHelper(allCandidates, currentJury, currentBias);
        currentJury.add(currentCandidate);
        findAllUnbiasedJuriesHelper(allCandidates, currentJury, currentBias + currentCandidate.bias);
        currentJury.remove(currentJury.size() - 1);

        allCandidates.insert(0, currentCandidate);
    }
}

void findAllUnbiasedJuries(Vector<juror>& allCandidates){
    Vector<juror> jury;
    findAllUnbiasedJuriesHelper(allCandidates, jury, 0, 0);
}
```

Jury Selection Code v2.0

```
void findAllUnbiasedJuriesHelper(Vector<juror>& allCandidates, Vector<juror>& currentJury, int
currentBias, int index){
    if (index == allCandidates.size()){
        if (currentBias == 0){
            displayJury(currentJury);
        }
    } else {
        juror currentCandidate = allCandidates[index];

        findAllUnbiasedJuriesHelper(allCandidates, currentJury, currentBias, index + 1);
        currentJury.add(currentCandidate);
        findAllUnbiasedJuriesHelper(allCandidates, currentJury, currentBias + currentCandidate.bias,
index + 1);
        currentJury.remove(currentJury.size() - 1);
    }
}

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    findAllUnbiasedJuriesHelper(allCandidates, jury, 0, 0);
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```

Jury Selection Code v2.0

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        findAllUnbiasedJuriesHelper(allCandidates, currentJury, currentBias + currentCandidate.bias,
index + 1);
        currentJury.remove(currentJury.size() - 1);
    }
}

void findAllUnbiasedJuries(Vector<juror>& allCandidates){
    Vector<juror> jury;
    findAllUnbiasedJuriesHelper(allCandidates, jury, 0, 0);
}
```

No more expensive addition/removal of possible candidates!

Takeaways

- Being able to enumerate all possible subsets and inspect subsets with certain constraints can be a powerful problem-solving tool.

- Maintaining an index of the current element under consideration for inclusion/exclusion in a collection is the most efficient way to keep track of the decision making process for subset generation

Jury Selection Code v2.0

```
void findAllUnbiasedJuriesHelper(Vector<juror>& allCandidates, Vector<juror>& currentJury, int
currentBias, int index){
    if (index == allCandidates.size()){
        if (currentBias == 0){
            displayJury(currentJury);
        }
    } else {
        juror currentCandidate = allCandidates[index];

        findAllUnbiasedJuriesHelper(allCandidates, currentJury, currentBias, index + 1);
        currentJury.add(currentCandidate);
        findAllUnbiasedJuriesHelper(allCandidates, currentJury, currentBias + currentCandidate.bias,
index + 1);
        currentJury.remove(currentJury.size() - 1);
    }
}

void findAllUnbiasedJuries(Vector<juror>& allCandidates){
    Vector<juror> jury;
    findAllUnbiasedJuriesHelper(allCandidates, jury, 0, 0);
}
```

Summary

Backtracking recursion:
Exploring many possible solutions

Overall paradigm: choose/explore/unchoose

Two ways of doing it

- **Choose explore undo**
  - Uses pass by reference; usually with large data structures
  - Explicit unchoose step by "undoing" prior modifications to structure
  - E.g. Generating subsets (one set passed around by reference to track subsets)
- **Copy edit explore**
  - Pass by value; usually when memory constraints aren't an issue
  - Implicit unchoose step by virtue of making edits to copy
  - E.g. Building up a string over time

Three use cases for backtracking

1. Generate/count all solutions (enumeration)
2. Find one solution (or prove existence)
3. Pick one best solution

General examples of things you can do:

- Permutations
- Subsets
- Combinations
- etc.

Goals for this Course

Learn how to model and solve complex problems with computers.

- Explore common abstractions for representing problems.
- Harness recursion and understand how to think about problems recursively.
- Analyze different approaches for solving problems: efficiency, optimization, and ethics.

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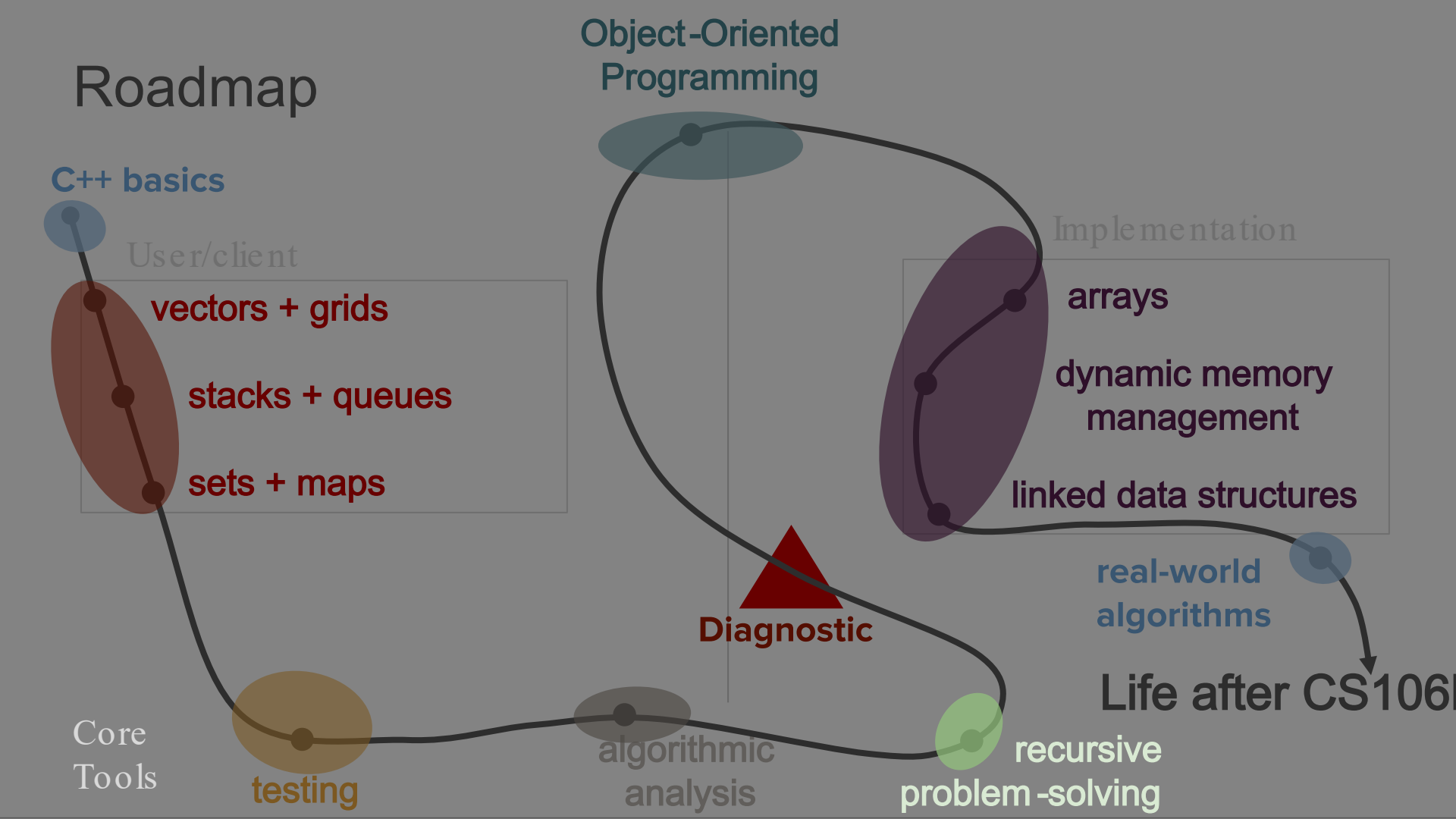
What’s next?


Roadmap
Object-Oriented Programming
C++ basics
User/client
vectors + grids
stacks + queues
sets + maps
Implementation
arrays
dynamic memory management
linked data structures
real-world algorithms
Diagnostic
Life after CS106
Core Tools
testing
algorithmic analysis
recursive problem-solving

More Recursive Backtracking