Combinatorics

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with materials by
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Combinatorics
Review: Course website

https://cs109.stanford.edu/
Logistics: Problem Set 1

14 questions (#1: tell me about yourself!)

Due: Wednesday, July 5 (before class)

Handwrite and scan

Word / Google Doc / ...

LaTeX 🎉

(see website for getting started!)
Logistics: Office hours

For SCPD: Thursday afternoon also online (Hangouts)!
Logistics: Textbook (or not)

Sheldon Ross
A First Course in Probability
9th (or 8th) Edition

Optional! Has lots of helpful examples.

Suggested readings on website.

Go up two floors for copies on reserve
(Terman Engineering Library)
Review: Principle of Inclusion/Exclusion

The **total number of elements** in two sets is the sum of the number of elements of each set, **minus** the number of elements in both sets.

\[ |A \cup B| = |A| + |B| - |A \cap B| \]

3 + 4 - 1 = 6
An experiment consisting of two or more separate parts has a number of outcomes equal to the product of the number of outcomes of each part.

$$|A_1 \times A_2 \times \cdots \times A_n| = \prod_i |A_i|$$

**Colors:** 3

**Shapes:** 4

Total: \(4 \cdot 3 = 12\)
Permutations

The number of ways of ordering $n$ distinguishable objects.

$$n! = 1 \cdot 2 \cdot 3 \cdot ... \cdot n = \prod_{i=1}^{n} i$$
Example: Servicing computers

9 computers to be scheduled for maintenance.

How many possible orders?
Example: Servicing computers

9 computers to be scheduled for maintenance.

How many possible orders?

Pick the first: 9 choices
Example: Servicing computers

9 computers to be scheduled for maintenance.

How many possible orders?

Pick the second: 8 choices
Example: Servicing computers

9 computers to be scheduled for maintenance.

How many possible orders?

9

8

Pick the third: 7 choices
Example: Servicing computers

9 computers to be scheduled for maintenance.

How many possible orders?
Example: Servicing computers

9 computers to be scheduled for maintenance.

How many possible orders?

9 × 8 × 7 × 6 × 5 × 4 × 3 × 2 × 1 = 362,880

9!
Example: Servicing computers

4 laptops, 3 tablets, 2 desktops. Same type have to be serviced together.

How many possible orders?
Example: Servicing computers

4 laptops, 3 tablets, 2 desktops.
Same type have to be serviced together.

How many possible orders?
Example: Servicing computers

4 laptops, 3 tablets, 2 desktops. Same type have to be serviced together.

How many possible orders?

\[4!\]
Example: Servicing computers

4 laptops, 3 tablets, 2 desktops. Same type have to be serviced together.

How many possible orders?

4! 3!
Example: Servicing computers

4 laptops, 3 tablets, 2 desktops. Same type have to be serviced together.

How many possible orders?

$4!$, $3!$, $2!$
Example: Servicing computers

4 laptops, 3 tablets, 2 desktops. Same type have to be serviced together.

How many possible orders?

\[ 4! \times 3! \times 2! = 288 \]
Example: Servicing computers

4 laptops, 3 tablets, 2 desktops. Same type have to be serviced together.

How many possible orders?

\[ 4! \times 3! \times 2! \times 3! = 1,728 \]
**binary:** Every node has at most two children.
Review: Binary search trees

**binary**: Every node has at most two children.

**search**: Root value is
- **greater** than values in the left subtree
- **less** than values in the right subtree
Degenerate binary search trees

degenerate: Every node has at most one child.
Degenerate binary search trees

degenerate: Every node has at most one child.

How many different BSTs containing the values 1, 2, and 3 are degenerate?
Degenerate binary search trees

BSTs can be listed by order of insertion.

How many possible orders for inserting 1, 2, and 3?
Degenerate binary search trees

BSTs can be listed by **order of insertion**.

How many possible orders for inserting 1, 2, and 3?

3! = 6

1, 2, 3  1, 3, 2  2, 1, 3  2, 3, 1  3, 1, 2  3, 2, 1
Degenerate binary search trees

BSTs can be listed by **order of insertion**.

How many possible orders for inserting 1, 2, and 3?

$$3! = 6$$

1, 2, 3  1, 3, 2  2, 1, 3  2, 3, 1  3, 1, 2  3, 2, 1
Degenerate binary search trees

BSTs can be listed by order of insertion.

How many possible orders for inserting 1, 2, and 3?

$$3! = 6$$

1, 2, 3   2, 3, 1   1, 3, 2   2, 1, 3   3, 1, 2   3, 2, 1
Degenerate binary search trees

BSTs can be listed by order of insertion.

How many possible orders for inserting 1, 2, and 3?

3! = 6

1, 2, 3  1, 3, 2  2, 1, 3  2, 3, 1  3, 1, 2  3, 2, 1
Degenerate binary search trees

BSTs can be listed by order of insertion.

How many possible orders for inserting 1, 2, and 3?

$3! = 6$

1, 2, 3  1, 3, 2  2, 1, 3  2, 3, 1  3, 1, 2  3, 2, 1
Degenerate binary search trees

BSTs can be listed by order of insertion.

How many possible orders for inserting 1, 2, and 3?

3! = 6

1, 2, 3  1, 3, 2  2, 1, 3  2, 3, 1  3, 1, 2  3, 2, 1
Degenerate binary search trees

BSTs can be listed by order of insertion.

How many possible orders for inserting 1, 2, and 3?

$3! = 6$

1, 2, 3  1, 3, 2  2, 1, 3  2, 3, 1  3, 1, 2  3, 2, 1
Degenerate binary search trees

BSTs can be listed by order of insertion.

How many possible orders for inserting 1, 2, and 3?

$3! = 6$

$1, 2, 3$, $1, 3, 2$, $2, 1, 3$, $2, 3, 1$, $3, 1, 2$, $3, 2, 1$
Degenerate binary search trees

BSTs can be listed by order of insertion.

How many possible orders for inserting 1, 2, and 3?

3! = 6

1, 2, 3  1, 3, 2  2, 1, 3  2, 3, 1  3, 1, 2  3, 2, 1

4 degenerate BSTs
Permutations

The number of ways of ordering $n$ distinguishable objects.

$$n! = 1 \cdot 2 \cdot 3 \cdots \cdot n = \prod_{i=1}^{n} i$$
Permutations with indistinct elements

The number of ways of ordering \( n \) objects, where some groups are indistinguishable.

\[
\binom{n}{k_1, k_2, \ldots, k_m} = \frac{n!}{k_1! \cdot k_2! \cdot \ldots \cdot k_m!}
\]
Return of the binary strings

How many **distinct** bit strings are there consisting of **three** 0's and **two** 1's?
Return of the binary strings

How many **distinct** bit strings are there consisting of **three** 0's and **two** 1's?

\[5! = 120\]
Return of the binary strings

How many distinct bit strings are there consisting of three 0's and two 1's?

\[5! = 120\]
Return of the binary strings

How many distinct bit strings are there consisting of three 0's and two 1's?

\[ \text{0 1 0 1 0} = 5! \]
Return of the binary strings

How many distinct bit strings are there consisting of three 0's and two 1's?

5!
Return of the binary strings

How many distinct bit strings are there consisting of three 0's and two 1's?

5!
Return of the binary strings

How many distinct bit strings are there consisting of three 0's and two 1's?

\[ 0 \ 1 \ 0 \ 1 \ 0 = 5! \]
Return of the binary strings

How many distinct bit strings are there consisting of three 0's and two 1's?

$$5! \times 2!$$
Return of the binary strings

How many distinct bit strings are there consisting of three 0's and two 1's?

\[
\begin{align*}
\text{5!} & \quad \cdot \quad \frac{2!}{3!} \\
& = 5!
\end{align*}
\]
Return of the binary strings

How many distinct bit strings are there consisting of three 0's and two 1's?

\[
\text{0 1 0 1 0} = 5! \cdot \frac{2!}{3!} \cdot \text{0 1 0 1 0}
\]
Return of the binary strings

How many distinct bit strings are there consisting of three 0's and two 1's?

\[ 5! = 2! \cdot 3! \cdot x \]
How many distinct bit strings are there consisting of three 0's and two 1's?

\[ x = \frac{5!}{2! \cdot 3!} \]
Example: Servicing computers

4 laptops, 3 tablets, 2 desktops.
Same type are indistinguishable.

How many possible orders?
Example: Servicing computers

4 laptops, 3 tablets, 2 desktops. Same type are indistinguishable.

How many possible orders?

9!
Example: Servicing computers
4 laptops, 3 tablets, 2 desktops. Same type are indistinguishable.

How many possible orders?

4! 9!
Example: Servicing computers

4 laptops, 3 tablets, 2 desktops. Same type are indistinguishable.

How many possible orders?

4! × 3! × 9!
Example: Servicing computers

4 laptops, 3 tablets, 2 desktops. Same type are indistinguishable.

How many possible orders?

4! x 3! x 2! x 9! = 9!
Example: Servicing computers

4 laptops, 3 tablets, 2 desktops. Same type are indistinguishable.

How many possible orders?

\[
\frac{9!}{4! \cdot 3! \cdot 2!} = 1,260
\]
Example: Servicing computers

4 laptops, 3 tablets, 2 desktops. Same type are indistinguishable.

How many possible orders?

\[
\binom{9}{4,3,2} = \frac{9!}{4! \cdot 3! \cdot 2!} = 1,260
\]
Example: Passcode guessing
4-digit passcode on a phone.

How many possible codes?

Room: CS109SUMMER17
Example: Passcode guessing

4-digit passcode on a phone.

How many possible codes?

\[ 10^4 = 10,000 \]

Room: CS109SUMMER17
Example: Passcode guessing

4 smudges on phone for a 4-digit passcode.

How many possible codes?

1
2
ABC
3
DEF
4
GHI
5
JKL
6
MNO
7
PQRS
8
TUV
9
WXYZ
0


Room: CS109SUMMER17
Example: Passcode guessing

4 smudges on phone for a 4-digit passcode.

How many possible codes?

4! = 24

Room: CS109SUMMER17
Example: Passcode guessing

3 smudges on phone for a 4-digit passcode.

How many possible codes?

1 2 3

1 DEF
4 5 6
4 JKL 6 MNO
7 8 9
7 PQRS 8 TUV 9 WXYZ
0

Room: CS109SUMMER17
Example: Passcode guessing

3 smudges on phone for a 4-digit passcode.

How many possible codes?

3 smudges = (less, same, more) possibilities vs. 4 smudges?

A) less  
B) same  
C) more

Room: CS109SUMMER17
Example: Passcode guessing

3 smudges on phone for a 4-digit passcode.

How many possible codes?

\[
\frac{4!}{2!1!1!} = 12
\]
Example: Passcode guessing

3 smudges on phone for a 4-digit passcode.

How many possible codes?

\[ 3 \cdot \frac{4!}{2!1!1!} = 36 \]
Example: Passcode guessing

2 smudges on phone for a 4-digit passcode.

How many possible codes?
Example: Passcode guessing

2 smudges on phone for a 4-digit passcode.

How many possible codes?

Two and two

Three and one
Example: Passcode guessing

2 smudges on phone for a 4-digit passcode.

How many possible codes?

Two and two
\[
\frac{4!}{2!2!} = 6
\]

Three and one
Example: Passcode guessing

2 smudges on phone for a 4-digit passcode.

How many possible codes?

Two and two
\[
\frac{4!}{2!2!} = 6
\]

Three and one
\[
\frac{2 \cdot 4!}{3!1!} = 8
\]
Example: Passcode guessing

2 smudges on phone for a 4-digit passcode.

How many possible codes?

Two and two
\[
\frac{4!}{2!2!} = 6
\]

Three and one
\[
\frac{2 \cdot 4!}{3!1!} = 8
\]

\[= 14\]
Permutations with indistinct elements

The number of ways of ordering $n$ objects, where some groups are indistinguishable.

$$\binom{n}{k_1, k_2, \ldots, k_m} = \frac{n!}{k_1! k_2! \ldots k_m!}$$
Break time!
Combinations

The number of unique subsets of size $k$ from a larger set of size $n$. (objects are distinguishable, unordered)

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}
\]
Picking drink flavors

5 drink flavors.
How many ways to pick 3?
Picking drink flavors

5 drink flavors.
How many ways to pick 3?

\[ \binom{5}{2} = \frac{5!}{2!3!} = 10 \]
Picking drink flavors

5 drink flavors.
How many ways to pick 3?

\[ \frac{5!}{3! \cdot 2!} = \binom{5}{3,2} \]
Return of the binary strings

How many distinct bit strings are there consisting of three 0's and two 1's?

\[ x = \frac{5!}{2! \cdot 3!} = \binom{5}{2,3} \]
Return of the binary strings

How many **distinct** bit strings are there consisting of **three** 0's and **two** 1's?

\[ x = \frac{5!}{2! \cdot 3!} = \binom{5}{2,3} \]

How many **subsets** of positions in a **five**-bit string are possible for placing **two** 1's?
Return of the binary strings

How many **distinct** bit strings are there consisting of **three** 0's and **two** 1's?

\[ x = \frac{5!}{2! \cdot 3!} = \binom{5}{2,3} = \binom{5}{2} \]

How many **subsets** of positions in a **five**-bit string are possible for placing **two** 1's?

\[ x = \frac{5!}{2! \cdot 3!} = \binom{5}{2,3} = \binom{5}{2} \]
Return of the binary strings

How many distinct bit strings are there consisting of three 0's and two 1's?

\[ x = \frac{5!}{3! \cdot 2!} = \binom{5}{3,2} = \binom{5}{3} \]

How many subsets of positions in a five-bit string are possible for placing three 0's?
Return of the binary strings

How many distinct bit strings are there consisting of three 0's and two 1's?

\[ x = \frac{5!}{3! \cdot 2!} = \binom{5}{3,2} = \binom{5}{3} = \binom{5}{2} \]

How many subsets of positions in a five-bit string are possible for placing three 0's?
Binomial coefficient

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}
\]
Combinations

The number of unique **subsets** of size $k$ from a larger set of size $n$.
(objects are distinguishable, unordered)

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}
\]
Bucketing

The number of ways of assigning $n$ distinguishable objects to a fixed set of $k$ buckets or labels.

$k^n$ n_objects

k_buckets
String hashing

$n = 3$ buckets

$m = 5$ strings

- "pacific cooler"
- "wild cherry"
  - "mountain cooler"
- "sugar"
  - "not sugar"
String hashing

$m = 5$ strings

$n = 3$ buckets

- “pacific cooler” 3
- “wild cherry” 3
- “mountain cooler” 3
- “sugar” 3
- “not sugar” 3

$= 3^5$
Divider method

The number of ways of assigning \( n \) indistinguishable objects to a fixed set of \( k \) buckets or labels.

\[
\binom{n + (k-1)}{n}
\]

\( n \) objects

\( k \) buckets

\( (k - 1 \text{ dividers}) \)
Indistinguishable drinks?
Indistinguishable drinks?
Indistinguishable drinks?
Indistinguishable drinks?
Indistinguishable drinks?
Indistinguishable drinks?

3 students → 3 - 1 = 2 dividers

5 drinks
Indistinguishable drinks?

7 objects
Indistinguishable drinks?

5 drinks

7 objects
Indistinguishable drinks?

5 drinks

7 objects
Indistinguishable drinks?

5 drinks

7 objects
Indistinguishable drinks?

\[
\binom{7}{5} = \frac{5 + (3 - 1)}{5} = 21
\]
Investing in startups
Investing in startups
Investing in startups
Investing in startups
Investing in startups
Investing in startups

\[
\begin{pmatrix}
10 + 3 \\
10
\end{pmatrix} = 286
\]
Permutations

The number of ways of ordering $n$ distinguishable objects.

$$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n = \prod_{i=1}^{n} i$$
Permutations with indistinct elements

The number of ways of ordering \( n \) objects, where some groups are indistinguishable.

\[
\left( \begin{array}{c} n \\ k_1, k_2, \ldots, k_m \end{array} \right) = \frac{n!}{k_1!k_2!\ldots k_m!}
\]
Combinations

The number of unique subsets of size $k$ from a larger set of size $n$. (objects are distinguishable, unordered)

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}
\]

choose $k$
Bucketing

The number of ways of assigning \( n \) distinguishable objects to a fixed set of \( k \) buckets or labels.

\[ k^n \]
Divider method

The number of ways of assigning $n$ indistinguishable objects to a fixed set of $k$ buckets or labels.

$\binom{n+(k-1)}{n}$

$n$ objects

$k$ buckets

$(k - 1$ dividers$)$