02: Combinatorics

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Adapted from slides by Lisa Yan
Takeaways from last time

**Inclusion-Exclusion Principle (generalized Sum Rule)**

If the outcome of an experiment can be either from Set $A$ or set $B$, where $A$ and $B$ may overlap, then the total number of outcomes of the experiment is

$$|A \cup B| = |A| + |B| - |A \cap B|.$$ 

**General Principle of Counting (generalized Product Rule)**

If an experiment has $r$ steps, such that step $i$ has $n_i$ outcomes for all $i = 1, \ldots, r$, then the total number of outcomes of the experiment is

$$n_1 \times n_2 \times \cdots \times n_r = \prod_{i=1}^{r} n_i.$$
### Essential information

<table>
<thead>
<tr>
<th>Website</th>
<th>cs109.stanford.edu</th>
</tr>
</thead>
</table>

#### Teaching Staff

[Image of teaching staff members]
Today’s plan

Permutations (sort objects)

Combinations (choose objects)

Put objects into buckets
Summary of Combinatorics

Counting tasks on \( n \) objects

- Sort objects (permutations)
  - Distinct (distinguishable): \( n! \)
  - Some distinct: \( \frac{n!}{n_1! n_2! \cdots n_r!} \)

- Choose \( k \) objects (combinations)
  - Distinct: \( \binom{n}{k} \)

- Put objects in \( r \) buckets
  - Distinct: \( \binom{n}{n_1, n_2, \ldots, n_r} \)
  - Indistinct: \( \frac{(n + r - 1)!}{n! (r - 1)!} \)

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Today’s plan

- Permutations (sort objects)
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  - Put objects into buckets
Summary of Combinatorics

Counting tasks on $n$ objects

- Sort objects (permutations)
- Choose $k$ objects (combinations)
- Put objects in $r$ buckets

Distinct (distinguishable)
Sort $n$ indistinct objects.
Sort $n$ distinct objects

Ayesha  Tim  Irina  Joey  Waddie

\[ 5 \times 4 \times 3 \times 2 \times 1 \]
Permutations

A permutation is an ordered arrangement of distinct objects.

The number of unique orderings (permutations) of $n$ distinct objects is

$$n! = n \times (n - 1) \times (n - 2) \times \cdots \times 2 \times 1.$$
Sort semi-distinct objects

All distinct

Ayesha  Tim  Irina  Joey  Waddie

Coca-Cola  Coca-Cola Zero  Coca-Cola  Coca-Cola  Coca-Cola

5! = 120

Some indistinct

Coke  Tim  Coke  Joey  Waddie

Coca-Cola  Coca-Cola Zero  Coca-Cola  Coca-Cola Zero  Coca-Cola

\[ \frac{5!}{2!} = \frac{120}{2} = 60 \]
Sort semi-distinct objects

How do we find the number of permutations considering some objects are indistinct?

By the product rule, permutations of distinct objects is a two-step process:

\[
\text{permutations of distinct objects} = \text{permutations considering some objects are indistinct} \times \text{Permutations of just the indistinct objects}
\]
Sort semi-distinct objects

How do we find the number of permutations considering some objects are indistinct?

By the product rule, permutations of distinct objects is a two-step process:

\[
\text{permutations of distinct objects} = \frac{\text{Permutations of just the indistinct objects}}{\text{permutations considering some objects are indistinct}}
\]
General approach to counting permutations

When there are $n$ objects such that

- $n_1$ are the same (indistinguishable or indistinct), and
- $n_2$ are the same, and
- ...
- $n_r$ are the same,

The number of unique orderings (permutations) is

$$\frac{n!}{n_1! \cdot n_2! \cdot \ldots \cdot n_r!}$$

For each group of indistinct objects,
Divide by the overcounted permutations
Sort semi-distinct objects

How many permutations?

\[
\frac{5!}{2!3!} = 10
\]
How many orderings of letters are possible for the following strings?

1. BOBA
   \[
   \frac{4!}{2! \cdot 1! \cdot 1!} = 12
   \]

2. MISSISSIPPI
   \[
   \frac{11!}{1! \cdot 4! \cdot 4! \cdot 2!} = 34,650
   \]
Summary of Combinatorics

Counting tasks on $n$ objects

- Sort objects (permutations)
  - Distinct (distinguishable)
    - $n!$
  - Some distinct
    - $\frac{n!}{n_1! n_2! \cdots n_r!}$

- Choose $k$ objects (combinations)

- Put objects in $r$ buckets
Today’s plan

Permutations (sort objects)

→ Combinations (choose objects)

Put objects into buckets
Summary of Combinatorics

Counting tasks on $n$ objects

- Sort objects (permutations): $n!$
- Choose $k$ objects (combinations):
  \[
  \frac{n!}{n_1! n_2! \cdots n_r!}
  \]
- Put objects in $r$ buckets: Distinct

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Combinations with cake

There are $n = 20$ people. How many ways can we choose $k = 5$ people to get cake?

Consider the following generative process...
Combinations with cake

There are \( n = 20 \) people. How many ways can we **choose** \( k = 5 \) people to get cake?

1. \( n \) people get in line

\( n! \) ways
Combinations with cake

There are $n = 20$ people. How many ways can we choose $k = 5$ people to get cake?

1. $n$ people get in line
2. Put first $k$ in cake room

$n!$ ways

1 way
Combinations with cake

There are \( n = 20 \) people.
How many ways can we choose \( k = 5 \) people to get cake?

1. \( n \) people get in line
   \( n! \) ways

2. Put first \( k \) in cake room
   1 way

\[ \text{overcounting} \quad 5! \]
Combinations with cake

There are \( n = 20 \) people. How many ways can we choose \( k = 5 \) people to get cake?

1. \( n \) people get in line
   - \( n! \) ways

2. Put first \( k \) in cake room
   - 1 way

3. Allow cake group to mingle
   - \( k! \) different permutations lead to the same mingle
Combinations with cake

There are $n = 20$ people. How many ways can we choose $k = 5$ people to get cake?

1. $n$ people get in line
   - $n!$ ways

2. Put first $k$ in cake room
   - 1 way

3. Allow cake group to mingle
   - $k!$ different permutations lead to the same mingle

4. Allow non-cake group to mingle

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Combinations with cake

There are $n = 20$ people.
How many ways can we choose $k = 5$ people to get cake?

1. $n$ people get in line
   $n!$ ways

2. Put first $k$ in cake room
   1 way

3. Allow cake group to mingle
   $k!$ different permutations lead to the same mingle

4. Allow non-cake group to mingle
   $(n - k)!$ different permutations lead to the same mingle
Combinations

A combination is an unordered selection of $k$ objects from a set of $n$ distinct objects.

The number of ways of making this selection is

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{k}
\]

**Interesting**

\[
\binom{n}{k} = \frac{n!}{(n-k)!(n-(n-k))!} = \frac{n!}{(n-k)!k!} = \binom{n}{k}
\]
Probability textbooks

1. How many ways are there to choose 3 books from a set of 6 distinct books?

\[ \binom{6}{3} = \frac{6!}{3!3!} = 20 \]
Probability textbooks

1. How many ways are there to choose 3 books from a set of 6 distinct books?
   \[
   \binom{6}{3} = \frac{6!}{3! \cdot 3!} = 20 \text{ ways}
   \]

2. What if we do not want to read both the 9th and 10th edition of Ross?

   Case 1: 9th ed + 2 other of 4
   \[
   \binom{4}{2} = 6
   \]

   Case 2: 10th ed + 2 other of 4
   \[
   \binom{4}{2} = 6
   \]

   \[
   \binom{4}{3} = 4
   \]

   \[
   \begin{pmatrix}
   \binom{4}{2} = 6 \\
   \binom{4}{3} = 4
   \end{pmatrix}
   \]

   Total: 16 ways
Probability textbooks (solution 2)

1. How many ways are there to choose 3 books from a set of 6 distinct books?

\[
\binom{6}{3} = \frac{6!}{3! 3!} = 20 \text{ ways}
\]

2. What if we do not want to read both the 9th and 10th edition of Ross?

**Forbidden Case:** 9th ed + 10th ed + 1 other book of 4

\[
\binom{4}{1} = 4
\]

So answer = 20 - 4 = 16

Sometimes easier to exclude forbidden cases.
Break
Announcements

**PS#1**
Out: today  
Due: Friday 1/17, 1:00pm  
Covers: through Friday

**Staff help**
Piazza policy: student discussion  
Office hours: start today  
[cs109.stanford.edu/staff.html](cs109.stanford.edu/staff.html)

**Python tutorial**
When: Friday 3:30-4:20pm  
Location: 420-041  
Recorded?: maybe  
Notes: to be posted online

**Section sign-ups**
Preference form: today  
Due: Saturday 1/11  
Results: latest Monday
### Geometric series:

\[ \sum_{i=0}^{n} x^i = \frac{1-x^{n+1}}{1-x} \]

\[ \sum_{i=m}^{n} x^i = \frac{x^{n+1}-x^m}{x-1} \]

\[ \sum_{i=0}^{\infty} x^i = \frac{1}{1-x} \text{ if } |x| < 1 \]

### Integration by parts (everyone’s favorite!):

Choose a suitable \( u \) and \( dv \) to decompose the integral of interest:

\[ \int u \cdot dv = u \cdot v - \int v \cdot du \]
Summary of Combinatorics

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- **Sort objects (permutations)**
  - Distinct (distinguishable): \( n! \)
  - Some distinct: \( \frac{n!}{n_1! n_2! \cdots n_r!} \)

- **Choose \( k \) objects (combinations)**
  - Distinct: \( \binom{n}{k} \)

- **Put objects in \( r \) buckets**
  - 1 group
  - \( r \) groups

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General approach to combinations

The number of ways to choose $r$ groups of $n$ distinct objects such that

For all $i = 1, \ldots, r$, group $i$ has size $n_i$, and $\Sigma_{i=1}^{r} n_i = n$ (all objects are assigned), is

\[
\frac{n!}{n_1! n_2! \ldots n_r!} = \binom{n}{n_1, n_2, \ldots, n_r}
\]

Multinomial Coefficient
Datacenters

13 different computers are to be allocated to 3 datacenters as shown in the table:

<table>
<thead>
<tr>
<th>Datacenter</th>
<th># machines</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>6</td>
</tr>
<tr>
<td>B</td>
<td>4</td>
</tr>
<tr>
<td>C</td>
<td>3</td>
</tr>
</tbody>
</table>

How many different divisions are possible?

\[
\binom{13}{6,4,3} = \frac{13!}{6!4!3!} = 60\,060
\]
Datacenters (solution 2)

13 different computers are to be allocated to 3 datacenters as shown in the table:

How many different divisions are possible?

Steps:
1. Choose 6 computers for A \( \binom{13}{6} \)
2. Choose 4 computers for B \( \binom{7}{4} \)
3. Choose 3 computers for C \( \binom{3}{3} \)

Choose \( k \) of \( n \) distinct objects into \( r \) groups of size \( n_1, \ldots, n_r \) \( \binom{n}{n_1, n_2, \ldots, n_r} \)

Datacenter | # machines
---|---
A | 6
B | 4
C | 3

\( \binom{13}{6} \binom{7}{4} \binom{3}{3} = 60,060 \)
Summary of Combinatorics

Counting tasks on \( n \) objects

- Sort objects (permutations)
  - Distinct (distinguishable)
    - \( n! \)
- Choose \( k \) objects (combinations)
  - Some distinct
    - \( \frac{n!}{n_1! n_2! \cdots n_r!} \)
  - Distinct
    - \( \binom{n}{k} \)
  - Same
    - 1 group
- Distinct
  - \( \binom{n}{n_1, n_2, \cdots, n_r} \)

Put objects in \( r \) buckets

Indistinct?

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A trick question

How many ways are there to group 6 indistinct (indistinguishable) objects into 3 groups, where group A, B, and C have size 1, 2, and 3, respectively?

Only 1

Choose $k$ of $n$ distinct objects into $r$ groups of size $n_1, \ldots, n_r$ \[ \binom{n}{n_1, n_2, \ldots, n_r} \]
Today’s plan

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- Sort objects (permutations)
  - Distinct (distinguishable)
    - $n!$
  - Some distinct
    - $\frac{n!}{n_1!n_2! \cdots n_r!}$
- Choose $k$ objects (combinations)
  - Distinct
    - $\binom{n}{k}$
  - Some distinct
    - $\binom{n}{n_1, n_2, \ldots, n_r}$
- Put objects in $r$ buckets
  - Distinct
  - Indistinct
Hash tables and **distinct** strings

How many ways are there to hash *n* **distinct** strings to *r* buckets?

**Steps:**

1. Bucket 1\(^{\text{st}}\) string
2. Bucket 2\(^{\text{nd}}\) string
   ...
3. Bucket *n*\(^{\text{th}}\) string

\[ \text{Total} = r^n \]
Summary of Combinatorics

Counting tasks on \( n \) objects

Sort objects (permutations)

- Distinct (distinguishable)
  \( n! \)

Choose \( k \) objects (combinations)

- Some distinct
  \( \frac{n!}{n_1!n_2! \cdots n_r!} \)
  \( \binom{n}{k} \)

- Distinct
  \( \binom{n}{n_1, n_2, \ldots, n_r} \)

Put objects in \( r \) buckets

- Distinct
  \( r^n \)

- Indistinct

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Hash tables and **indistinct** strings

How many ways are there to distribute $n$ **indistinct** strings to $r$ buckets?

**Goal**

Bucket 1 has $x_1$ strings,
Bucket 2 has $x_2$ strings,
...
Bucket $r$ has $x_r$ strings (the rest)

*How many different sets of counts are possible?*
Simple example: \( n = 3 \) strings and \( r = 2 \) buckets

<table>
<thead>
<tr>
<th>sss</th>
<th>ss/s</th>
<th>s/s</th>
<th>s/s/s</th>
</tr>
</thead>
</table>

All permutations of three S and are divider:

\[
\frac{4!}{3! \cdot 1!} = \binom{4}{1} = 4
\]
Bicycle helmet sales

How many ways can we assign $n = 5$ indistinguishable children to $r = 4$ distinct bicycle helmet styles?

Consider the following generative process...
Bicycle helmet sales: 1 possible assignment outcome

How many ways can we assign \( n = 5 \) indistinguishable children to \( r = 4 \) distinct bicycle helmet styles?

\( n = 5 \) indistinct objects \hspace{1cm} r = 4 \) distinct buckets
Bicycle helmet sales: 1 possible assignment outcome

How many ways can we assign \( n = 5 \) indistinguishable children to \( r = 4 \) distinct bicycle helmet styles?

\[
\begin{align*}
n &= 5 \text{ indistinct objects} & r &= 4 \text{ distinct buckets}
\end{align*}
\]

Goal: Order \( n \) indistinct objects and \( r - 1 \) indistinct dividers.
Bicycle helmet sales: A generative proof

How many ways can we assign $n = 5$ indistinguishable children to $r = 4$ distinct bicycle helmet styles?

**Goal** Order $n$ indistinct objects and $r - 1$ indistinct dividers.

0. Make objects and dividers distinct
Bicycle helmet sales: A generative proof

How many ways can we assign \( n = 5 \) indistinguishable children to \( r = 4 \) distinct bicycle helmet styles?

**Goal** Order \( n \) indistinct objects and \( r - 1 \) indistinct dividers.

0. Make objects and dividers distinct

1. Order \( n \) distinct objects and \( r - 1 \) distinct dividers
   \[ (n + r - 1)! \]
Bicycle helmet sales: A generative proof

How many ways can we assign $n = 5$ indistinguishable children to $r = 4$ distinct bicycle helmet styles?

**Goal** Order $n$ indistinct objects and $r - 1$ indistinct dividers.

0. Make objects and dividers distinct

1. Order $n$ distinct objects and $r - 1$ distinct dividers
   
   \[
   (n + r - 1)!
   \]

2. Make $n$ objects indistinct
   
   \[
   \frac{1}{n!}
   \]
Bicycle helmet sales: A generative proof

How many ways can we assign \( n = 5 \) indistinguishable children to \( r = 4 \) distinct bicycle helmet styles?

**Goal**  Order \( n \) indistinct objects and \( r - 1 \) indistinct dividers.

0. Make objects and dividers distinct

1. Order \( n \) distinct objects and \( r - 1 \) distinct dividers

\[(n + r - 1)!\]

2. Make \( n \) objects indistinct

\[\frac{1}{n!}\]

3. Make \( r - 1 \) dividers indistinct

\[\frac{1}{(r - 1)!}\]
Divider method

The number of ways to distribute $n$ indistinct objects into $r$ buckets is equivalent to the number of ways to permute $n + r - 1$ objects such that $n$ are indistinct objects, and $r - 1$ are indistinct dividers:

$$\text{Total} = \frac{(n+r-1)!}{n!} \cdot \frac{1}{(r-1)!} = \binom{n+r-1}{r-1}$$
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Sort objects (permutations)

- Distinct (distinguishable): \( n! \)
- Some distinct: \( \frac{n!}{n_1! n_2! \cdots n_r!} \)

Choose \( k \) objects (combinations)

- Distinct: \( \binom{n}{k} \)
- 1 group: \( \binom{n}{n_1, n_2, \ldots, n_r} \)
- \( r \) groups: \( \binom{n}{n_1, n_2, \ldots, n_r} \)

Put objects in \( r \) buckets

- Distinct: \( r^n \)
- Indistinct: \( \frac{(n + r - 1)!}{n! (r - 1)!} \)