Today we will make history
Logistics: Office hours
Review: Permutations

The number of ways of ordering $n$ distinguishable objects.

$$n! = 1 \cdot 2 \cdot 3 \cdots \cdot n = \prod_{i=1}^{n} i$$
Review: Combinations

The number of unique \textbf{subsets} of size $k$ from a larger set of size $n$. (objects are distinguishable, unordered)

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}
\]
Review: Bucketing

The number of ways of assigning $n$ distinguishable objects to a fixed set of $k$ buckets or labels.

$k^n$
Divider method

The number of ways of assigning \( n \) indistinguishable objects to a fixed set of \( k \) buckets or labels.

\[
\binom{n + (k - 1)}{n}
\]

\( n \) objects

<p>| | | | |</p>
<table>
<thead>
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</table>
\( k \) buckets

(k - 1 dividers)
A grid of ways of counting

<table>
<thead>
<tr>
<th></th>
<th>Ordering</th>
<th>Subsets</th>
<th>Bucketing</th>
</tr>
</thead>
<tbody>
<tr>
<td>All distinct</td>
<td>$n!$</td>
<td>(n \choose k)</td>
<td>$k^n$</td>
</tr>
<tr>
<td>Some indistinct</td>
<td>$\frac{n!}{k_1! k_2! \ldots k_m!}$</td>
<td>Creativity!</td>
<td></td>
</tr>
</tbody>
</table>
| - Split into cases
| - Use inclusion/exclusion
| - Reframe the problem |
| All indistinct | 1          | 1         | $\binom{n + (k - 1)}{n}$ |
Sample space

$S =$ the set of all possible outcomes

- Coin flip: $S = \{\text{Heads, Tails}\}$
- Two coin flips: $S = \{(H, H), (H, T), (T, H), (T, T)\}$
- Roll of 6-sided die: $S = \{1, 2, 3, 4, 5, 6\}$
- # emails in day: $S = \mathbb{N}$ (non-neg. integers)
- Netflix hours in day: $S = [0, 24]$ (interval, inclusive)
Events

\[ E = \text{some subset of } S \quad (E \subseteq S) \]

- Coin flip is heads \( E = \{\text{Heads}\} \)
- \( \geq 1 \) head on 2 flips \( E = \{(H, H), (H, T), (T, H)\} \)
- Roll of die \( \leq 3 \) \( E = \{1, 2, 3\} \)
- \# emails in day \( \leq 20 \) \( E = \{x \mid x \in \mathbb{N}, x \leq 20\} \)
- "Wasted day" \( E = [5, 24] \)

(Ross uses \( \subset \) to mean \( \subseteq \))
Set operations

Union: outcomes that are in E or F

\[ E = \{1, 2\} \]
\[ F = \{2, 3\} \]

\[ S = \{1, 2, 3, 4, 5, 6\} \]

\[ E \cup F = \{1, 2, 3\} \]
Set operations

Intersection: outcomes that are in E and F

$S = \{1, 2, 3, 4, 5, 6\}$

$E = \{1, 2\}$

$F = \{2, 3\}$

$EF = E \cap F = \{2\}$
Set operations

Complement: outcomes that are not in E

\[ S = \{1, 2, 3, 4, 5, 6\} \]

\[ E = \{1, 2\} \]

\[ F = \{2, 3\} \]

\[ E^c = \{3, 4, 5, 6\} \]
A profound truth

A profound truth

Everything in the world is either

a hot dog

or

not a hot dog.

\( (E \cup E^c = S) \)

Quiz question

https://b.socrative.com/login/student/

Room: CS109SUMMER17
Quiz question

Which is the correct picture for $E^c F^c (= E^c \cap F^c)$?
Quiz question

Which is the correct picture for $E^c F^c (= E^c \cap F^c)$?
Quiz question

Which is the correct picture for $E^c \cup F^c$?
Quiz question

Which is the correct picture for $E^c \cup F^c$?
De Morgan's Laws

“distributive laws” for set complement

\[ (E \cup F)^c = E^c \cap F^c \]

\[ (E \cap F)^c = E^c \cup F^c \]

\[ \left( \bigcup_{i} E_i \right)^c = \bigcap_{i} E_i^c \]

\[ \left( \bigcap_{i} E_i \right)^c = \bigcup_{i} E_i^c \]
What is a probability?

A number between 0 and 1 to which we ascribe meaning
Sources of probability

- Experiment
- Datasets
- Expert opinion
- Analytical solution
Meaning of probability

A quantification of ignorance
What is a probability?

\[ P(E) = \lim_{n \to \infty} \frac{\#(E)}{n} \]
Axioms of probability

(1) \[ 0 \leq P(E) \leq 1 \]

(2) \[ P(S) = 1 \]

(3) If \( E \cap F = \emptyset \), then
\[ P(E \cup F) = P(E) + P(F) \]
(Sum rule, but with probabilities!)
Corollaries

\[ P(E^c) = 1 - P(E) \]

If \( E \subseteq F \), then \( P(E) \leq P(F) \)

\[ P(E \cup F) = P(E) + P(F) - P(EF) \]

(Principle of inclusion/exclusion, but with probabilities!)
Principle of Inclusion/Exclusion

The **total number of elements** in two sets is the sum of the number of elements of each set, **minus** the number of elements in both sets.

\[ |A \cup B| = |A| + |B| - |A \cap B| \]

\[ 3 + 4 - 1 = 6 \]
Inclusion/exclusion with more than two sets

\[ |E \cup F \cup G| = \]
Inclusion/exclusion with more than two sets

\[ |E \cup F \cup G| = |E| \]
Inclusion/exclusion with more than two sets

\[ |E \cup F \cup G| = |E| + |F| \]
Inclusion/exclusion with more than two sets

\[ |E \cup F \cup G| = |E| + |F| + |G| \]
Inclusion/exclusion with more than two sets

\[ |E \cup F \cup G| = |E| + |F| + |G| - |EF| \]
Inclusion/exclusion with more than two sets

\[ |E \cup F \cup G| = |E| + |F| + |G| - |EF| - |EG| \]
Inclusion/exclusion with more than two sets

\[ |E \cup F \cup G| = |E| + |F| + |G| \]
\[ - |EF| - |EG| - |FG| \]
Inclusion/exclusion with more than two sets

\[ |E \cup F \cup G| = |E| + |F| + |G| - |EF| - |EG| - |FG| + |EFG| \]
Inclusion/exclusion with more than two sets

$\left| \bigcup_{i=1}^{n} E_i \right| = \sum_{r=1}^{n} (-1)^{r+1} \sum_{i_1 < \cdots < i_r} \left| \bigcap_{j=1}^{r} E_{i_j} \right|$
Inclusion/exclusion with more than two sets

\[
P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{r=1}^{n} \left(-1\right)^{r+1} \sum_{i_1 < \ldots < i_r} P\left(\bigcap_{j=1}^{r} E_{i_j}\right)
\]

prob. of OR
sum over subset sizes
add or subtract (based on size)
prob. of AND
sum over all subsets of that size
Inclusion/exclusion with more than two sets

\[ P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(EF) - P(EG) - P(FG) + P(EFG) \]
Break time!
Equally likely outcomes

Coin flip \[ S = \{ \text{Heads, Tails} \} \]

Two coin flips \[ S = \{ (H, H), (H, T), (T, H), (T, T) \} \]

Roll of 6-sided die \[ S = \{ 1, 2, 3, 4, 5, 6 \} \]

\[ P(\text{Each outcome}) = \frac{1}{|S|} \]

\[ P(E) = \frac{|E|}{|S|} \quad \text{(counting!)} \]
Rolling two dice

\[ P(D_1 + D_2 = 7) = ? \]
How do I get started?

For word problems involving probability, start by defining events!
Rolling two dice

$D_1$  $D_2$

$P(D_1 + D_2 = 7) = ?$

$E$: {all outcomes such that the sum of the two dice is 7}
Rolling two dice

\[ P(D_1 + D_2 = 7) = \frac{6}{36} = \frac{1}{6} \]

\[ S = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), \boxed{(1, 6)}, \]
\[ (2, 1), (2, 2), (2, 3), (2, 4), \boxed{(2, 5)}, (2, 6), \]
\[ (3, 1), (3, 2), (3, 3), \boxed{(3, 4)}, (3, 5), (3, 6), \]
\[ (4, 1), (4, 2), \boxed{(4, 3)}, (4, 4), (4, 5), (4, 6), \]
\[ (5, 1), \boxed{(5, 2)}, (5, 3), (5, 4), (5, 5), (5, 6), \]
\[ \boxed{(6, 1)}, (6, 2), (6, 3), (6, 4), (6, 5), (6, 6) \} \]
Rolling two (indistinguishable) dice?

\[ P(D_1 + D_2 = 7) = \frac{3}{21} = \frac{1}{7} \]

\[ S = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 3), (3, 4), (3, 5), (3, 6), (4, 4), (4, 5), (4, 6), (5, 5), (5, 6), (6, 6)\} \]
Getting rid of ORs

Finding the probability of an OR of events can be nasty. Try using De Morgan's laws to turn it into an AND!

\[ P(A \cup B \cup \cdots \cup Z) = 1 - P(A^c B^c \cdots Z^c) \]
Running into a friend

21,000 people at Stanford
You're friends with 250

Go to an event with 70 other (equally-likely) Stanforders

What's $P(\text{at least one friend among the 70})$?

$|E| = ???$

$|S| = \binom{21,000}{70}$

$E$: \{subsets of size 70 with at least one friend\}

$S$: \{all subsets of size 70 of 21,000 students\}
Running into a friend

21,000 people at Stanford
You're friends with 250

Go to an event with 70 other (equally-likely) Stanforders

What's \( P(\text{at least one friend among the 70}) \)?:

\[
\begin{align*}
|E^c| &= \binom{21,000 - 250}{70} \\
|S| &= \binom{21,000}{70}
\end{align*}
\]

\[
P(E) = 1 - P(E^c) = 1 - \frac{\binom{20,750}{70}}{\binom{21,000}{70}} \approx 0.5512
\]

\( E \): \{subsets of size 70 with at least one friend\}

\( E^c \): \{subsets of size 70 with no friends\}

\( S \): \{all subsets of size 70 of 21,000 students\}
Running into a friend


**serendipity** [ˌsərənˈdɪ.pɪ.ri] *n* the faculty or phenomenon of finding valuable or agreeable things not sought for
Getting rid of ORs

Finding the probability of an OR of events can be nasty. Try using De Morgan's laws to turn it into an AND!

\[ P(A \cup B \cup \cdots \cup Z) = 1 - P(A^c B^c \cdots Z^c) \]
Defective chips

\[ |S| = \binom{n}{k} \]

\[ |E| = 1 \cdot \binom{n-1}{k-1} \]

**E:** \{all outcomes such that the defective chip is chosen\}

**S:** \{all subsets of size \( k \) from \( n \) chips\}
Defective chips

\[ |E| = 1 \cdot \binom{n-1}{k-1} \]

\[ |S| = \binom{n}{k} \]

\[
P(E) = \frac{n-1}{k} = \frac{(n-1)!}{(k-1)!(n-k)!} = \frac{(n-1)!k!}{n!(k-1)!} = \frac{n-1)! \cdot k \cdot (k-1)!}{n \cdot (n-1)! (k-1)!} = \frac{k}{n}
\]
Poker straight

(Standard deck:)
52 cards = 13 ranks × 4 suits

“Straight”: 5 consecutive ranks
(suits can be different)

\[ |E| = 10 \cdot 4^5 \]

\[ |S| = \binom{52}{5} \]

\( \text{starting rank} \)

\( \text{suit of each card} \)

\( \text{E: \{all hands that are straights\}} \)

\( \text{S: \{all possible hands\}} \)
Poker straight

(Standard deck:)
52 cards = 13 ranks × 4 suits

“Straight”: 5 consecutive ranks
(suits can be different)

\[ |E| = 10 \cdot 4^5 \]
\[ |S| = \binom{52}{5} \]

\[ P(E) = \frac{10 \cdot 4^5}{\binom{52}{5}} \approx 0.00394 \]
Birthdays

\( n \) people in a room.
What is the probability none share the same birthday?

\( E: \{\text{all ways of giving all } n \text{ people different birthdays}\} \)

\( S: \{\text{all ways of giving all } n \text{ people each a birthday}\} \)

\(|E| = 365 \cdot 364 \cdot \ldots \cdot (365-n+1) = \frac{365!}{(365-n)!} = \binom{365}{n} \cdot n! \)

\(|S| = 365^n \)

\[ P(E) = \frac{\binom{365}{n} \cdot n!}{365^n} \]

\( n = 23: \quad P(E) < 1/2 \)
\( n = 75: \quad P(E) < 1/3,000 \)
\( n = 100: \quad P(E) < 1/3,000,000 \)
\( n = 150: \quad P(E) < 1/3,000,000,000,000,000,000,000 \)
Flipping cards

- Shuffle deck.
- Reveal cards from the top until we get an Ace. Put Ace aside.
- What is \( P(\text{next card is the Ace of Spades}) \)?
- \( P(\text{next card is the 2 of Clubs}) \)?

A) \( P(\text{Ace of Spades}) > P(\text{2 of Clubs}) \)

B) \( P(\text{Ace of Spades}) = P(\text{2 of Clubs}) \)

C) \( P(\text{Ace of Spades}) < P(\text{2 of Clubs}) \)
Flipping cards

- Shuffle deck.
- Reveal cards from the top until we get an Ace. Put Ace aside.
- What is $P(\text{next card is the Ace of Spades})$?
- $P(\text{next card is the 2 of Clubs})$?

$E$: {all orderings for which Ace of Spades comes immediately after first Ace}

$S$: {all orderings of the deck}

$|E| = 51! \cdot 1$

$|S| = 52!$

$P(E) = \frac{51!}{52!} = \frac{1}{52}$
Flipping cards

- Shuffle deck.
- Reveal cards from the top until we get an Ace. Put Ace aside.
- What is $P(\text{next card is the Ace of Spades})$?
- $P(\text{next card is the 2 of Clubs})$?

$E$: \{all orderings for which 2 of Clubs comes immediately after first Ace\}

$S$: \{all orderings of the deck\}

$|E| = 51! \cdot 1$

$|S| = 52!$

$P(E) = \frac{51!}{52!} = \frac{1}{52}$
Flipping cards

- Shuffle deck.
- Reveal cards from the top until we get an Ace. Put Ace aside.
- What is $P(\text{next card is the Ace of Spades})$?
- $P(\text{next card is the 2 of Clubs})$?

**B** $P(\text{Ace of Spades}) = P(\text{2 of Clubs})$
(We just made history)