Lecture 03: Intro to Probability

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June 29, 2018
Announcements

PS1: due Friday 7/6
  ◦ Covers material up to and including today

Python tutorial on website

Office hours have started!
  ◦ SCPD-supported hours are marked on calendar
Summary of Combinatorics

Counting operations on \( n \) objects

- **Ordered permutations**
  - Distinguishable
    - \( n! \)
  - Some indistinguishable
    - \( \frac{n!}{n_1!n_2!\ldots} \)

- **Choose \( k \) combinations**
  - Distinguishable
    - \( \binom{n}{k} \)
  - Indistinguishable
    - \( \binom{n}{n_1, n_2, \ldots, n_r} \)

- **Put in \( r \) buckets**
  - Distinguishable
    - \( r^n \)
  - Indistinguishable
    - \( \frac{(n + r - 1)!}{n!(r - 1)!} \)
DNA distance

For a DNA tree we need to calculate the DNA distance between each pair of animals. How many calculations are needed?

How many distinct pairs of \( n \) animals are there?

\[
\binom{n}{2} = \frac{n^2 - n}{2}
\]
Goals for today

Probability
- Sample spaces and events
- Axioms of Probability
- Corollaries of Axioms of Probability
- Equally Likely Outcomes

"Started from the bottom now we’re here" – Drake, 2013
Sample spaces

The **sample space**, $S$, is the set of all possible outcomes of an experiment.

Examples:

- **Coin flip:** $S = \{\text{Head, Tails}\}$
- **Flipping two coins:** $S = \{(H, H), (H, T), (T, H), (T, T)\}$
- **Roll of 6-sided die:** $S = \{1, 2, 3, 4, 5, 6\}$
- **# emails in a day:** $S = \{x \mid x \in \mathbb{Z}, x \geq 0\}$ (non-negative integers)
- **YouTube hrs. in day:** $S = \{x \mid x \in \mathbb{R}, 0 \leq x \leq 24\}$
Events

An event, $E$, is some subset of $S$ ($E \subseteq S$).

Examples:
- Coin flip is heads: $E = \{\text{Head}\}$
- $\geq 1$ head on 2 coin flips: $E = \{(H, H), (H, T), (T, H)\}$
- Roll of die is 3 or less: $E = \{1, 2, 3\}$
- # emails in a day $\leq 20$: $E = \{x \mid x \in \mathbb{Z}, 0 \leq x \leq 20\}$
- Wasted day ($\geq 5$ YT hrs.): $E = \{x \mid x \in \mathbb{R}, 5 \leq x \leq 24\}$
Set operations on events

Say that $E$ and $F$ are events in $S$. 
Union of events

Define the new event $E \cup F$ as the union of events $E$ and $F$, which contains all outcomes that are in $E$ or $F$.

- $S = \{1, 2, 3, 4, 5, 6\}$ die roll outcome
- $E = \{1, 2\}$
- $F = \{2, 3\}$
- $E \cup F = \{1, 2, 3\}$
Intersection of events

Define the new event $E \cap F$ as the intersection of events $E$ and $F$, which contains all outcomes that are in both $E$ and $F$.

- $S = \{1, 2, 3, 4, 5, 6\}$ die roll outcome
- $E = \{1, 2\}$ $F = \{2, 3\}$
- $E \cap F = \{2\}$

Note: mutually exclusive events means that $E \cap F = EF = \emptyset$
Complement of an event

Define the new event $E^c$ the complement of the event $E$, which contains all outcomes that are not in $E$.

- $S = \{1, 2, 3, 4, 5, 6\}$ die roll outcome
- $E = \{1, 2\}$
- $E^c = \{3, 4, 5, 6\}$
DeMorgan’s Laws

\[(E \cup F)^c = E^c \cap F^c\]

\[(E \cap F)^c = E^c \cup F^c\]

\[\left( \bigcup_{i=1}^{n} E_i \right)^c = \bigcap_{i=1}^{n} E_i^c\]

\[\left( \bigcap_{i=1}^{n} E_i \right)^c = \bigcup_{i=1}^{n} E_i^c\]

General form
What is a probability?
Definition of Probability

A number between 0 and 1 to which we ascribe meaning.
Definition of Probability

\[ P(E) = \lim_{n \to \infty} \frac{n(E)}{n} \]

* Our belief that an event E occurs.
Axioms of Probability

Axiom 1: \( 0 \leq P(E) \leq 1 \)

Axiom 2: \( P(S) = 1 \)

Axiom 3: If \( E \) and \( F \) mutually exclusive \( (E \cap F = \emptyset) \), then \( P(E) + P(F) = P(E \cup F) \)

For any sequence of mutually exclusive events \( E_1, E_2, \ldots \)

\[
P\left( \bigcup_{i=1}^{\infty} E_i \right) = \sum_{i=1}^{\infty} P(E_i) \quad \text{(like the Sum Rule of Counting, but for probabilities)}
\]
Corollaries of Axioms

1. \( P(E^c) = 1 - P(E) \)  
   \((= P(S) - P(E) )\)

2. If \( E \subseteq F \), then \( P(E) \leq P(F) \)

3. \( P(E \cup F) = P(E) + P(F) - P(EF) \)  
   Inclusion-Exclusion Principle for Probability

General form of Inclusion-Exclusion Identity:

\[
P \left( \bigcup_{i=1}^{n} E_i \right) = \sum_{r=1}^{n} (-1)^{r+1} \sum_{i_1 < \ldots < i_r} P \left( \bigcap_{j=1}^{r} E_{i_j} \right)
\]
General form of Inclusion-Exclusion

Include each space exactly once.

\[ P(E \cup F \cup G) \]

\[ r = 1 \]

\[ = P(E) + P(F) + P(G) \]

\[ r = 2 \]

\[ - P(E \cap F) - P(E \cap G) - P(F \cap G) \]

\[ + P(E \cap F \cap G) \]

\[ P \left( \bigcup_{i=1}^{n} E_i \right) = \sum_{r=1}^{n} (-1)^{r+1} \sum_{i_1 < \ldots < i_r} P \left( \bigcap_{j=1}^{r} E_{i_j} \right) \]
Selecting Programmers

- $P(\text{student programs in Java}) = 0.28 = P(A)$
- $P(\text{student programs in Python}) = 0.07 = P(B)$
- $P(\text{student programs in Java and Python}) = 0.05. = P(A \cap B)$

What is $P(\text{student does not program in (Java or Python)})$?

Solution:
Define: $A = \text{event that student programs in Java}$
$B = \text{event that student programs in Python}$

Want to find (WTF): $P((A \cup B)^c) = 1 - P(A \cup B)$

$= 1 - [P(A) + P(B) - P(A \cap B)]$

$= 1 - (0.28 + 0.07 - 0.05) = 0.7 \rightarrow 70\%$

$P(\text{student programs in Python, but not Java})$?

$P(A^c \cap B) = P(B) - P(A \cap B) = 0.07 - 0.05 = 0.02 \rightarrow 2\%$
Equally Likely Outcomes

Some sample spaces have equally likely outcomes

- Coin flip: \( S = \{ \text{Head, Tails} \} \)
- Flipping two coins: \( S = \{ (H, H), (H, T), (T, H), (T, T) \} \)
- Roll of 6-sided die: \( S = \{ 1, 2, 3, 4, 5, 6 \} \)

\[
P(\text{Each outcome}) = \frac{1}{|S|}
\]

In that case, \( P(E) = \frac{\text{number of outcomes in } E}{\text{number of outcomes in } S} = \frac{|E|}{|S|} \) (by Axiom 3 of probability)
Rolling two dice

Problem:
Roll two 6-sided dice. What is \( P(\text{sum} = 7) \)?

Solution:
\[
S = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6),
(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6),
(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6),
(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6),
(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6),
(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}
\]
\[
E = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}
\]
\[
\frac{6}{36} = \frac{1}{6}
\]
Rolling two dice the wrong way

Problem:
Roll two 6-sided dice. What is $P(\text{sum} = 7)$?

Wrong Solution:

More likely \hspace{1cm} Less likely

11 outcomes for rolling a sum: \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}

7 is only one of these outcomes, so $\frac{1}{11}$ (wrong)
Break!

Attendance: tinyurl.com/cs109summer2018
Cats and carrots

Problem:

4 cats and 3 carrots in a bag. 3 drawn.

What is \( P(1 \text{ cats and } 2 \text{ carrots drawn}) \)?

Consider:

What is the sample space that will give you equally likely outcomes?
Cats and carrots

Problem:
4 cats and 3 carrots in a bag. 3 drawn.
What is P(1 cat and 2 carrots drawn)?

Solution 1:
Define sample space: Ordered list of 3 distinguishable objects

\[ |S| = 7 \times 6 \times 5 = 210 \]

Valid events:

\[ |E| = (4 \times 3 \times 2) + (3 \times 4 \times 2) + (3 \times 2 \times 4) = 72 \]

(\text{cat as #1}) \quad (\text{cat as #2}) \quad (\text{cat as #3})

P(1 cat, 2 carrots) = |E|/|S| = 72/210 = 12/35
Cats and carrots

Problem:
4 cats and 3 carrots in a bag. 3 drawn.

What is $P(1$ cats and $2$ carrots drawn)?

Solution 2:
Define sample space:

$$|S| = \binom{7}{3} = 35$$

Valid events:

$$|E| = \binom{4}{1}\binom{3}{2} = 12$$

(choose the cat)  (choose the 2 carrots)

Unordered set of 3 distinguishable objects

$$P(1 \text{ cats, 2 carrots}) = \frac{|E|}{|S|} = \frac{12}{35}$$
You often make indistinguishable items distinguishable in order to get equally likely sample space outcomes.
Chip Defect Detection

$n$ chips are manufactured, 1 of which is defective. $k$ chips are randomly selected from $n$ for testing.

What is $P$(defective chip is in $k$ selected chips?)

Solution:

$|S| = \binom{n}{k}$

$|E| = \binom{1}{1}\binom{n-1}{k-1}$

$P($defective chip in $k$ selected chips$) = P(E)$

$$= \frac{(n-1)}{\binom{k-1}{k}} = \frac{(n-1)!}{(k-1)!(n-k)!} = \frac{n!}{k!(n-k)!} = \frac{(n-1)!k!}{n!(k-1)!} = \frac{k}{n}$$
Chip Defect Detection

Problem:
n chips are manufactured, 1 of which is defective. k chips are randomly selected from n for testing.
What is P(defective chip is in k selected chips?)

Solution:
1. Choose k chips.
2. Throw a dart and make one defective.
P(defective one in k selected chips) = \frac{k}{n}
Any Straight in Poker

Problem:
Consider 5 card poker hands.

- “straight” is 5 consecutive rank cards of any suit
- What is $P(\text{straight})$?

Solution:

$|S| = \binom{52}{5}$

$|E| = 10 \binom{4}{1}^5$

$P(\text{straight}) = \frac{10 \binom{4}{1}^5}{\binom{52}{5}} \approx 0.00394$
“Official” Straight in Poker

Problem:
Consider 5 card poker hands.

• “straight” is 5 consecutive rank cards of any suit
• “straight flush” is 5 consecutive rank cards of same suit
• What is \( P(\text{straight but not straight flush}) \)?

Solution:
\[
|S| = \binom{52}{5}
\]
\[
|E| = 10\binom{4}{1}^5 - 10\binom{4}{1}
\]

\[
P(\text{straight but not straight flush}) = \frac{10\binom{4}{1}^5 - 10\binom{4}{1}}{\binom{52}{5}} \approx 0.00392
\]
Card Flipping

In a 52 card deck, cards are flipped one at a time. After the first ace (of any suit) appears, consider the next card.

Is $P(\text{next card} = \text{Ace Spades}) < P(\text{next card} = \text{2 Clubs})$?

Solution:

You might think so initially, but...

$|S| = 52!$ (shuffling cards)

Case 1: $E_{AS} = \text{next card is Ace Spades}$
1. Take out Ace of Spades.
2. Shuffle leftover 51 cards.
3. Add Ace Spades after first ace.

$|E_{AS}| = 51! \cdot 1$

$P(E_{AS}) = P(E_{AS})$

Case 2: $E_{2C} = \text{next card is Two Clubs}$
1. Take out 2 Clubs.
2. Do the same thing as Case 1

$|E_{2C}| = 51! \cdot 1$
It is often easier to calculate $P(E^c)$.

DeMorgan’s: \[ P(A \cup B \cup \cdots \cup Z) = 1 - P(A^c B^c \cdots Z^c) \]
The Birthday Paradox Problem

What is the probability that in a set of $n$ people, at least one pair of them will share the same birthday?

Solution:

$P(\text{at least one pair shares birthday}) = P(E)$

$\quad = 1 - P(\text{no one shares a birthday}) = 1 - P(E^c)$

$P(\text{no one shares a birthday}) = P(E^c)$:

$|E^c| = \{\text{all ways that } n \text{ people can have different birthdays}\}$

$\quad = (365)(364)\ldots(365 - n + 1) = \binom{365}{n}n!$

$|S| = \{\text{all ways } n \text{ people can have any birthdays}\} = 365^n$

$P(\text{no matching birthdays}) = \frac{|E^c|}{|S|} = \frac{\binom{365}{n} \cdot n!}{365^n}$
The Birthday Paradox Problem

What is the probability that in a set of \( n \) people, at least one pair of them will share the same birthday?

Solution:

\[
\begin{align*}
P(\text{no matching birthdays}) &= P(E^c) = \frac{|E^c|}{|S|} = \frac{\binom{365}{n} \cdot n!}{365^n} \\
\Rightarrow P(\text{at least one pair shares a birthday}) &= 1 - P(\text{no one shares a birthday}) \\
 &= 1 - \frac{\binom{365}{n} \cdot n!}{365^n} \\
\end{align*}
\]

- \( n = 23: \) \( P(E) > \frac{1}{2} \)
- \( n = 75: \) \( P(E) > 1 - 1/3,000 = 0.9997 \)
- \( n = 100: \) \( P(E) > 1 - 1/3,000,000 = 0.9999997 \)
- \( n = 150: \) \( P(E) > 1 - 1/3,000,000,000,000,000 \)

By Pigeonhole principle, we would need 366 people to reach 100%!
The Harder Birthday Problem

Problem:
What is the probability that of \( n \) other people, someone has the same birthday *as you*?

Solution:
\[
P(\text{someone has your birthday}) = P(E) = 1 - P(E^c)
\]
\[
|E^c| = \{\text{all ways that no one has your birthday}\} = 364^n
\]
\[
|S| = 365^n
\]
\[
P(E) = 1 - \frac{364^n}{365^n}
\]

\( n = 23: \quad P(E) \approx 0.0612 \)
\( n = 150: \quad P(E) \approx 0.3374 \)
\( n = 253: \quad P(E) \approx 0.5005 \)

Why are these probabilities much higher than before?
- Anyone born on June 22\(^{nd}\)?
- Is today anyone’s birthday?
Summary

Sample space, S: The set of all possible outcomes of an experiment.
Event space, E: A subset of S.
If we have equally likely outcomes, then \( P(E) = \frac{|E|}{|S|} \).

Two key tactics to counting probabilities:
- We can treat indistinct objects as distinct if it helps create equally likely outcomes.
- DeMorgan’s law and calculating \( P(E^c) \) helps us avoid tricky counting situations.
Individual Problems with Chance:

Any problems with Chance:

Event $E$: no problems

$$E = (P_1 \cup P_2 \cup \ldots \cup P_n)^c$$

Don’t want no problems:

$$E^c = ((P_1 \cup P_2 \cup \ldots \cup P_n)^c)^c$$

$$= P_1 \cup P_2 \cup \ldots \cup P_n$$

We want problems with you, Chance!!!

"You don’t want no problems, want no problems with me”
– Chance the Rapper, 2016