13: Independent RVs

Lisa Yan
October 21, 2019
Probabilities from joint CDFs

Joint CDF: \( P(X \leq x, Y \leq y) = F_{X,Y}(x, y) \)

\[
P(a_1 < X \leq a_2, b_1 < Y \leq b_2) = \]
\[
F_{X,Y}(a_2, b_2) - F_{X,Y}(a_1, b_2) - F_{X,Y}(a_2, b_1) + F_{X,Y}(a_1, b_1)
\]
Probabilities from joint CDFs

Joint CDF: \( P(X \leq x, Y \leq y) = F_{X,Y}(x, y) \)

\[
P(a_1 < X \leq a_2, b_1 < Y \leq b_2) = F_{X,Y}(a_2, b_2) - F_{X,Y}(a_1, b_2) - F_{X,Y}(a_2, b_1) + F_{X,Y}(a_1, b_1)
\]
Probabilities from joint CDFs

Joint CDF: \( P(X \leq x, Y \leq y) = F_{X,Y}(x, y) \)

\[
P(a_1 < X \leq a_2, b_1 < Y \leq b_2) = F_{X,Y}(a_2, b_2) - F_{X,Y}(a_1, b_2) - F_{X,Y}(a_2, b_1) + F_{X,Y}(a_1, b_1)
\]
Joint CDF: $P(X \leq x, Y \leq y) = F_{X,Y}(x, y)$

$$P(a_1 < X \leq a_2, b_1 < Y \leq b_2) = F_{X,Y}(a_2, b_2) - F_{X,Y}(a_1, b_2) - F_{X,Y}(a_2, b_1) + F_{X,Y}(a_1, b_1)$$
Probabilities from joint CDFs

Joint CDF: \( P(X \leq x, Y \leq y) = F_{X,Y}(x, y) \)

\[
P(a_1 < X \leq a_2, b_1 < Y \leq b_2) = \]
\[
F_{X,Y}(a_2, b_2) - F_{X,Y}(a_1, b_2) - F_{X,Y}(a_2, b_1) + F_{X,Y}(a_1, b_1)
\]
Probabilities from joint CDFs

Joint CDF: \( P(X \leq x, Y \leq y) = F_{X,Y}(x, y) \)

\[
P(a_1 < X \leq a_2, b_1 < Y \leq b_2) = F_{X,Y}(a_2, b_2) - F_{X,Y}(a_1, b_2) - F_{X,Y}(a_2, b_1) + F_{X,Y}(a_1, b_1)
\]
Probabilities from joint CDFs

Joint CDF: $P(X \leq x, Y \leq y) = F_{X,Y}(x, y)$

$$P(a_1 < X \leq a_2, b_1 < Y \leq b_2) = F_{X,Y}(a_2, b_2) - F_{X,Y}(a_1, b_2) - F_{X,Y}(a_2, b_1) + F_{X,Y}(a_1, b_1)$$
Probability with Instagram!

In image processing, a Gaussian blur is the result of blurring an image by a Gaussian function. It is a widely used effect in graphics software, typically to reduce image noise.

\[
P(a_1 < X \leq a_2, b_1 < Y \leq b_2) = F_{X,Y}(a_2, b_2) - F_{X,Y}(a_1, b_2) - F_{X,Y}(a_2, b_1) + F_{X,Y}(a_1, b_1)
\]
Gaussian blur

In a Gaussian blur, for every pixel:

• Weight each pixel by the probability that \( X \) and \( Y \) are both within the pixel bounds
• The weighting function is a Gaussian joint PDF with a standard deviation parameter \( \sigma \).

Gaussian blurring with \( \sigma = 3 \)

Joint PDF:
\[
    f_{X,Y}(x, y) = \frac{1}{2\pi \cdot 3^2} e^{-\left(\frac{x^2 + y^2}{2 \cdot 3^2}\right)}
\]

Joint CDF:
\[
    F_{X,Y}(x, y) = \Phi \left(\frac{x}{3}\right) \Phi \left(\frac{y}{3}\right)
\]

Weight matrix:

Center pixel: (0, 0)

Pixel bounds:
\[-0.5 < x \leq 0.5\]
\[-0.5 < y \leq 0.5\]
Gaussian blur

In a Gaussian blur:

• Weight each pixel by the probability that $X$ and $Y$ are both within the pixel bounds

What is the weight of the center pixel?

$$P(-0.5 < X \leq 0.5, -0.5 < Y \leq 0.5)$$
$$= F_{X,Y}(0.5,0.5) - F_{X,Y}(-0.5,0.5) - F_{X,Y}(0.5,-0.5) + F_{X,Y}(-0.5,-0.5)$$
$$= \Phi \left( \frac{0.5}{3} \right) \Phi \left( \frac{0.5}{3} \right) - 2 \cdot \Phi \left( \frac{-0.5}{3} \right) \Phi \left( \frac{0.5}{3} \right)$$
$$+ \Phi \left( \frac{-0.5}{3} \right) \Phi \left( \frac{-0.5}{3} \right)$$
$$\approx 0.5662^2 - 2 \cdot 0.5662 \cdot 0.4338 + 0.4338^2$$
$$\approx 0.206$$

Gaussian blurring with $\sigma = 3$

Joint PDF:

$$f_{X,Y}(x,y) = \frac{1}{2\pi \cdot 3^2} e^{-(x^2+y^2)/2 \cdot 3^2}$$

Joint CDF:

$$F_{X,Y}(x, y) = \Phi \left( \frac{x}{3} \right) \Phi \left( \frac{y}{3} \right)$$

Weight matrix:

Center pixel: $(0, 0)$

Pixel bounds:

$$-0.5 < x \leq 0.5$$
$$-0.5 < y \leq 0.5$$
**CS109 roadmap**

Multiple events:

<table>
<thead>
<tr>
<th>Intersection</th>
<th>Conditional probability</th>
<th>Independence</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(E \cap F)$</td>
<td>$P(E</td>
<td>F) = \frac{P(EF)}{P(F)}$</td>
</tr>
</tbody>
</table>

Joint (Multivariate) distributions

- **Joint PMF/PDF**
  - $p_{X,Y}(x,y)$
  - $f_{X,Y}(x,y)$

- **Conditional distributions?**
  - Yes! (Wednesday)

- **Independent RVs?**
  - Yes! (today)

Model ALL the things!
Today’s plan  (covered on midterm)

Independent RVs

Sum of independent RVs
• ✔ Binomial
• ✔ Convolution
• ✔ Poisson
• ✔ Normal
• ⚠ Uniform

Expectation of sum of RVs (next class)
Independent discrete RVs

Recall the definition of independent events $E$ and $F$:

$$P(EF) = P(E)P(F)$$

Two discrete random variables $X$ and $Y$ are independent if:

$$P(X = x, Y = y) = P(X = x)P(Y = y)$$

Different notation, same idea:

$$p_{X,Y}(x, y) = p_X(x)p_Y(y)$$

Intuitively: knowing value of $X$ tells us nothing about the distribution of $Y$ (and vice versa)

If two variables are not independent, they are called dependent.
Dice (after all this time, still our friends)

Let: $D_1$ and $D_2$ be the outcomes of two rolls

$S = D_1 + D_2$, the sum of two rolls

- Each roll of a 6-sided die is an independent trial.
- $D_1$ and $D_2$ are independent.

Are $S$ and $D_1$ independent?

1. $P(D_1 = 1, S = 7)$?
2. $P(D_1 = 1, S = 5)$?
Dice (after all this time, still our friends)

Let: \( D_1 \) and \( D_2 \) be the outcomes of two rolls
\[ S = D_1 + D_2, \text{ the sum of two rolls} \]

- Each roll of a 6-sided die is an independent trial.
- \( D_1 \) and \( D_2 \) are independent.

Are \( S \) and \( D_1 \) independent? ❌

1. \( P(D_1 = 1, S = 7) \)?
   
   Event \((S = 7)\): \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}
   
   \[ P(D_1 = 1)P(S = 7) = (1/6)(1/6) = 1/36 = P(D_1 = 1, S = 7) \]

   **Independent** events \((D_1 = 1), (S = 7)\)

2. \( P(D_1 = 1, S = 5) \)?
   
   Event \((S = 5)\): \{(1,4), (2,3), (3,2), (4,1)\}
   
   \[ P(D_1 = 1)P(S = 5) = (1/6)(4/36) \neq 1/36 = P(D_1 = 1, S = 5) \]

All events \((X = x, Y = y)\) must be independent for \(X, Y\) to be independent random variables.
Coin flips

Flip a coin with probability $p$ of “heads” a total of $n + m$ times.

Let $X = \text{number of heads in first } n \text{ flips. } X \sim \text{Bin}(n, p)$
$Y = \text{number of heads in next } m \text{ flips. } Y \sim \text{Bin}(m, p)$
$Z = \text{total number of heads in } n + m \text{ flips.}$

1. Are $X$ and $Z$ independent? ❌

Counterexample: What if $Z = 0$?
Coin flips

Flip a coin with probability \( p \) of “heads” a total of \( n + m \) times.

Let \( X = \) number of heads in first \( n \) flips. \( X \sim \text{Bin}(n, p) \)
\( Y = \) number of heads in next \( m \) flips. \( Y \sim \text{Bin}(m, p) \)
\( Z = \) total number of heads in \( n + m \) flips.

1. Are \( X \) and \( Z \) independent? \( \times \) \hspace{1cm} Counterexample: What if \( Z = 0 \)?

2. Are \( X \) and \( Y \) independent?

Strategy:
A. No, proof by counterexample
B. Yes, proof by counting
C. None/other
Coin flips

Flip a coin with probability $p$ of “heads” a total of $n + m$ times.

Let $X =$ number of heads in first $n$ flips. $X \sim \text{Bin}(n, p)$
$Y =$ number of heads in next $m$ flips. $Y \sim \text{Bin}(m, p)$
$Z =$ total number of heads in $n + m$ flips.

1. Are $X$ and $Z$ independent? ❌
   Counterexample: What if $Z = 0$?

2. Are $X$ and $Y$ independent? ✓

Strategy:

A. No, proof by counterexample
B. Yes, proof by counting
C. None/other
Coin flips

Flip a coin with probability $p$ of “heads” a total of $n + m$ times.

Let 
\[ X = \text{number of heads in first } n \text{ flips. } X \sim \text{Bin}(n, p) \]
\[ Y = \text{number of heads in next } m \text{ flips. } Y \sim \text{Bin}(m, p) \]
\[ Z = \text{total number of heads in } n + m \text{ flips.} \]

1. Are $X$ and $Z$ independent? ❌

2. Are $X$ and $Y$ independent? ✓

\[
P(X = x, Y = y) = P \left( \begin{array}{c} \text{first } n \text{ flips have } x \text{ heads} \\ \text{and next } m \text{ flips have } y \text{ heads} \end{array} \right)
\]
\[
= \binom{n}{x} p^x (1 - p)^{n-x} \binom{m}{y} p^y (1 - p)^{m-y}
\]
\[
= P(X = x)P(Y = y)
\]
Independent continuous RVs

Two continuous random variables $X$ and $Y$ are independent if:

$$P(X \leq x, Y \leq y) = P(X \leq x)P(Y \leq y)$$

Equivalently:

$$F_{X,Y}(x, y) = F_X(x)F_Y(y)$$
$$f_{X,Y}(x, y) = f_X(x)f_Y(y)$$

More generally, $X$ and $Y$ are independent if joint density factors separately:

$$f_{X,Y}(x, y) = h(x)g(y), \text{ where } -\infty < x, y < \infty$$
Is the Gaussian blur distribution independent?

Gaussian blurring with $\sigma = 3$

Joint PDF:
$$f_{X,Y}(x, y) = \frac{1}{2\pi \cdot 3^2} e^{-\frac{(x^2 + y^2)}{2 \cdot 3^2}}$$

Joint CDF:
$$F_{X,Y}(x, y) = \Phi \left(\frac{x}{3}\right) \Phi \left(\frac{y}{3}\right)$$

Weight matrix:

Center pixel: (0, 0)
Pixel bounds:
$$-0.5 < x \leq 0.5$$
$$-0.5 < y \leq 0.5$$
Independent continuous RVs

Two continuous random variables $X$ and $Y$ are independent if:

$$P(X \leq x, Y \leq y) = P(X \leq x)P(Y \leq y)$$

Equivalently:

$$F_{X,Y}(x, y) = F_X(x)F_Y(y)$$
$$f_{X,Y}(x, y) = f_X(x)f_Y(y)$$

More generally, $X$ and $Y$ are independent if joint density factors separately:

$$f_{X,Y}(x, y) = g(x)h(y), \text{ where } -\infty < x, y < \infty$$
Pop quiz! (just kidding)

Are $X$ and $Y$ independent in the following cases?

1. $f_{X,Y}(x, y) = 6e^{-3x}e^{-2y}$
   where $0 < x, y < \infty$

2. $f_{X,Y}(x, y) = 4xy$
   where $0 < x, y < 1$

3. $f_{X,Y}(x, y) = 4xy$
   where $0 < x + y < 1$
Pop quiz! (just kidding)

Are $X$ and $Y$ independent in the following cases?

1. $f_{X,Y}(x, y) = 6e^{-3x}e^{-2y}$
   where $0 < x, y < \infty$
   Separable functions:
   $g(x) = 3e^{-3x}$
   $h(y) = 2e^{-2y}$
   independent $X$ and $Y$

2. $f_{X,Y}(x, y) = 4xy$
   where $0 < x, y < 1$
   Separable functions:
   $g(x) = 2x$
   $h(y) = 2y$

3. $f_{X,Y}(x, y) = 4xy$
   where $0 < x + y < 1$
   Cannot capture constraint on $x + y$
   into factorization!

If you can factor densities over all of the support, you have independence.
Break for jokes/announcements
Announcements

Midterm exam

When: Tuesday, October 29th, 7:00pm-9:00pm
Where: Hewlett 200
Covers: Up to (and including) week 4 + Lecture Notes #13
Review session: Saturday, 10am-12pm, Shiram 104

Problem Set 4

Out: later today
Due: Wednesday 11/6
Midterm coverage: First half (marked)

Concept checks

Week 4’s: Tuesday 10/22 1pm
Week 5’s: Wednesday 10/31 1pm
Today’s plan

Independent RVs

Sum of independent RVs

• ✅ Binomial
• Convolution
• ✅ Poisson
• ✅ Normal
• ⚠️ Uniform

Expectation of sum of RVs (next class)
Sum of independent Binomials

\[ X \sim \text{Bin}(n_1, p) \]
\[ Y \sim \text{Bin}(n_2, p) \]
\[ X, Y \text{ independent} \]

\[ X + Y \sim \text{Bin}(n_1 + n_2, p) \]

Intuition:

- Each trial in \( X \) and \( Y \) is independent and has same success probability \( p \)
- Define \( Z = n_1 + n_2 \) independent trials, each with success probability \( p \)
  \( Z \sim \text{Bin}(n_1 + n_2, p) \), and also \( Z = X + Y \)

Holds in general case:

\[ X_i \sim \text{Bin}(n_i, p) \]
\[ X_i \text{ independent for } i = 1, \ldots, n \]

\[ \sum_{i=1}^{n} X_i \sim \text{Bin} \left( \sum_{i=1}^{n} n_i, p \right) \]
Convolution: Sum of independent random variables

For any discrete random variables $X$ and $Y$:

$$P(X + Y = n) = \sum_k P(X = k, Y = n - k)$$

In particular, for independent discrete random variables $X$ and $Y$:

$$P(X + Y = n) = \sum_k P(X = k)P(Y = n - k)$$

the convolution of $p_X$ and $p_Y$
Insight into convolution

For independent discrete random variables $X$ and $Y$:

$$P(X + Y = n) = \sum_{k} P(X = k)P(Y = n - k)$$

Suppose $X$ and $Y$ are independent, both with support $\{0, 1, \ldots\}$:

<table>
<thead>
<tr>
<th>$X = k$</th>
<th>$Y = n - k$</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$n$</td>
<td>$P(X = 0)P(Y = n)$</td>
</tr>
<tr>
<td>1</td>
<td>$n - 1$</td>
<td>$P(X = 1)P(Y = n - 1)$</td>
</tr>
<tr>
<td>2</td>
<td>$n - 2$</td>
<td>$P(X = 2)P(Y = n - 2)$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$n$</td>
<td>0</td>
<td>$P(X = n)P(Y = 0)$</td>
</tr>
<tr>
<td>$n + 1$</td>
<td>–</td>
<td>0</td>
</tr>
</tbody>
</table>

Sum of mutually exclusive events
The distribution of a sum of dice rolls is a convolution.

Note for $k, n − k$ in the support,

\[
P(X = k, Y = n − k) = P(X = k)P(Y = n − k) = 1/36
\]
Today’s plan

Independent RVs

Sum of independent RVs
  • ✅ Binomial
  • Convolution
  • ✅ Poisson
  • ✅ Normal
  • ⚠️ Uniform

Expectation of sum of RVs (next class)
Sum of independent Poissons

\[ X \sim \text{Poi}(\lambda_1), \ Y \sim \text{Poi}(\lambda_2) \]
\[ X, Y \text{ independent} \]

\[ X + Y \sim \text{Poi}(\lambda_1 + \lambda_2) \]

Proof (just for reference):

\[
P(X + Y = n) = \sum_{k=0}^{n} P(X = k)P(Y = n - k)
\]

\[
= \sum_{k=0}^{n} e^{-\lambda_1} \frac{\lambda_1^k}{k!} e^{-\lambda_2} \frac{\lambda_2^{n-k}}{(n-k)!} = e^{-(\lambda_1+\lambda_2)} \sum_{k=0}^{n} \frac{\lambda_1^k \lambda_2^{n-k}}{k!(n-k)!}
\]

\[
= e^{-(\lambda_1+\lambda_2)} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \frac{\lambda_1^k \lambda_2^{n-k}}{\lambda_1 + \lambda_2}^{n} = e^{-(\lambda_1+\lambda_2)} \frac{n!}{n!} (\lambda_1 + \lambda_2)^n
\]

\[ \text{PMF of Poisson RVs} \]

\[ X \text{ and } Y \text{ independent, convolution} \]

\[ \text{Binomial Theorem:} \]

\[(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \]
General sum of independent Poissons

Holds in general case:

\[ X_i \sim \text{Poi}(\lambda_i) \]
\[ X_i \text{ independent for } i = 1, \ldots, n \]

\[ \sum_{i=1}^{n} X_i \sim \text{Poi}\left( \sum_{i=1}^{n} \lambda_i \right) \]
### Sum of independent Gaussians

**Formula:**

\[ X \sim \mathcal{N}(\mu_1, \sigma_1^2), \quad Y \sim \mathcal{N}(\mu_2, \sigma_2^2) \]

\[ X + Y \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2) \]

**Independence:**

\[ X, Y \text{ independent} \]


Holds in general case:

\[ X_i \sim \mathcal{N}(\mu_i, \sigma_i^2) \]

\[ X_i \text{ independent for } i = 1, \ldots, n \]

\[ \sum_{i=1}^{n} X_i \sim \mathcal{N} \left( \sum_{i=1}^{n} \mu_i , \sum_{i=1}^{n} \sigma_i^2 \right) \]
Virus infections

Suppose you are working with the WHO to plan a response to the initial conditions of a virus. There are two exposed groups:

- G1: 200 people, each independently infected with $p_1 = 0.1$
- G2: 100 people, each independently infected with $p_2 = 0.4$

What is $P(\text{people infected } \geq 55)$?

1. Define RVs & state goal

   Let $A = \# \text{ infected in G1}$. \\
   $A \sim \text{Bin}(200, 0.1)$ \\
   $B = \# \text{ infected in G2}$. \\
   $B \sim \text{Bin}(100, 0.4)$

   Want: $P(A + B \geq 55)$

   Strategy:
   A. Dance, Dance, Convolution
   B. Sum of indep. Binomials
   C. (approximate) Sum of indep. Poissons
   D. (approximate) Sum of indep. Normals
   E. None/other
Virus infections

Suppose you are working with the WHO to plan a response to the initial conditions of a virus. There are two exposed groups:

- G1: 200 people, each independently infected with $p_1 = 0.1$
- G2: 100 people, each independently infected with $p_2 = 0.4$

What is $P$(people infected $\geq 55$)?

1. Define RVs & state goal

Let $A = \#$ infected in G1.
   $A \sim \text{Bin}(200, 0.1)$

Let $B = \#$ infected in G2.
   $B \sim \text{Bin}(100, 0.4)$

Want: $P(A + B \geq 55)$

Strategy:

A. Dance, Dance, Convolution
B. Sum of indep. Binomials
C. (approximate) Sum of indep. Poissons
D. (approximate) Sum of indep. Normals
E. None/other
Virus infections

Suppose you are working with the WHO to plan a response to the initial conditions of a virus. There are two exposed groups:

- G1: 200 people, each independently infected with $p_1 = 0.1$
- G2: 100 people, each independently infected with $p_2 = 0.4$

What is $P$(people infected $\geq 55$)?

1. Define RVs & state goal

Let $A = \#$ infected in G1.

\[ A \sim \text{Bin}(200, 0.1) \]

Let $B = \#$ infected in G2.

\[ B \sim \text{Bin}(100, 0.4) \]

Want: $P(A + B \geq 55)$

2. Approximate as sum of Normals

\[ A \approx X \sim \mathcal{N}(20, 18) \quad B \approx Y \sim \mathcal{N}(40, 24) \]

\[ P(A + B \geq 55) \approx P(X + Y \geq 54.5) \]

3. Solve

continuity correction
Virus infections

Suppose you are working with the WHO to plan a response to the initial conditions of a virus. There are two exposed groups:

- G1: 200 people, each independently infected with $p_1 = 0.1$
- G2: 100 people, each independently infected with $p_2 = 0.4$

What is $P(\text{people infected } \geq 55)$?

1. Define RVs & state goal
   Let $A = \# \text{ infected in G1}$.
   $A \sim \text{Bin}(200,0.1)$
   $B = \# \text{ infected in G2}$.
   $B \sim \text{Bin}(100,0.4)$
   Want: $P(A + B \geq 55)$

2. Approximate as sum of Normals
   
   $A \approx X \sim \mathcal{N}(20,18)$
   $B \approx Y \sim \mathcal{N}(40,24)$

   
   $P(A + B \geq 55) \approx P(X + Y \geq 54.5)$

   (continuity correction)

3. Solve
   Let $W = X + Y \sim \mathcal{N}(20 + 40 = 60, 18 + 24 = 42)$
   
   $P(W \geq 54.5) = 1 - \Phi \left( \frac{54.5 - 60}{\sqrt{42}} \right) \approx 1 - \Phi(-0.85)$

   $\approx 0.8023$
Linear transforms vs. independence

Let $X \sim \mathcal{N}(\mu, \sigma^2)$ and $Y = X + X$. What is the distribution of $Y$?

- Are both approaches valid?

**Independent RVs approach**

Let $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$, $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$ be independent.
Then $Y = X_1 + X_2 \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

**Linear transform approach**

Let $X \sim \mathcal{N}(\mu, \sigma^2)$.
If $Y = aX + b$,
then $Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$. 
Linear transforms vs. independence

Let $X \sim \mathcal{N}(\mu, \sigma^2)$ and $Y = X + X$. What is the distribution of $Y$?

- Are both approaches valid?

**Independent RVs approach**

Let $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2), X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$ be independent.

Then $Y = X_1 + X_2 \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

**Linear transform approach**

Let $X \sim \mathcal{N}(\mu, \sigma^2)$.

If $Y = aX + b$, then $Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$.

$Y = X + X$

$X + X \sim \mathcal{N}(\mu + \mu, \sigma^2 + \sigma^2)$ **$X$ is NOT independent of $X$!**

$Y \sim \mathcal{N}(2\mu, 2\sigma^2)$?

$Y = 2X$

$Y \sim \mathcal{N}(2\mu, 4\sigma^2)$
Motivating idea: Zero sum games

Want: \( P(\text{Warriors win}) = P(A_W > A_B) \)
\[ = P(A_W - A_B > 0) \]

Assume \( A_W, A_B \) are independent.
Let \( D = A_W - A_B \).

What is the distribution of \( D \)?

A. \( D \sim \mathcal{N}(1657 - 1470, 200^2 - 200^2) \)
B. \( D \sim \mathcal{N}(1657 - 1470, 200^2 + 200^2) \)
C. \( D \sim \mathcal{N}(1657 + 1470, 200^2 + 200^2) \)
D. Dance, Dance, Convolution
E. None/other
Motivating idea: Zero sum games

Want: \( P(\text{Warriors win}) = P(A_W > A_B) \)
\[ = P(A_W - A_B > 0) \]

Assume \( A_W, A_B \) are independent.
Let \( D = A_W - A_B \).

What is the distribution of \( D \)?

A. \( D \sim \mathcal{N}(1657 - 1470, 200^2 - 200^2) \)
B. \( D \sim \mathcal{N}(1657 - 1470, 200^2 + 200^2) \)
C. \( D \sim \mathcal{N}(1657 + 1470, 200^2 + 200^2) \)
D. Dance, Dance, Convolution
E. None/other

If \( X \sim \mathcal{N}(\mu_1, \sigma^2) \),
then \((-X) \sim \mathcal{N}(-\mu, (-1)^2 \sigma^2 = \sigma^2)\)
Motivating idea: Zero sum games

Want: \( P(\text{Warriors win}) = P(A_W > A_B) \)
\[ = P(A_W - A_B > 0) \]

Assume \( A_W, A_B \) are independent.
Let \( D = A_W - A_B \).
\[ D \sim \mathcal{N}(1657 - 1470, 200^2 + 200^2) \]
\[ \sim \mathcal{N}(187, 2 \cdot 200^2) \quad \sigma \approx 283 \]

\[ P(D > 0) = 1 - F_D(0) = 1 - \Phi \left( \frac{0 - 187}{283} \right) \]
\[ \approx 0.7454 \]

Compare with 0.7488, calculated by sampling!
Today’s plan

Independent RVs

Sum of independent RVs
• ✅ Binomial
• ✅ Convolution
• ✅ Poisson
• ✅ Normal
• ⚠ Uniform

Expectation of sum of RVs (next class)
Dance, Dance, Convolution Extreme

For independent discrete random variables $X$ and $Y$:

\[
P(X + Y = n) = \sum_k P(X = k)P(Y = n - k)
\]

the convolution of $p_X$ and $p_Y$

For independent continuous random variables $X$ and $Y$:

\[
f_{X+Y}(\alpha) = \int_{-\infty}^{\infty} f_X(x)f_Y(\alpha - x)\,dx
\]

the convolution of $f_X$ and $f_Y$
Sum of independent Uniforms

Let $X \sim \text{Uni}(0,1)$ and $Y \sim \text{Uni}(0,1)$ be independent random variables. What is the distribution of $X + Y$, $f_{X+Y}$?

$$f_{X+Y}(\alpha) = \int_{-\infty}^{\infty} f_X(k)f_Y(\alpha - k)dk$$

$f_X(k)f_Y(\alpha - k) = 1$ when: (select one)

A. between 0 and 1
B. $0 \leq k \leq 1$
C. $0 \leq \alpha - k \leq 1$
D. $0 \leq \alpha \leq 2$
E. Other

$X$ and $Y$ independent + continuous

$f_{X+Y}(\alpha) = \int_{-\infty}^{\infty} f_X(x)f_Y(\alpha - x) \, dx$
Sum of independent Uniforms

Let \( X \sim \text{Uni}(0,1) \) and \( Y \sim \text{Uni}(0,1) \) be independent random variables. What is the distribution of \( X + Y \), \( f_{X+Y} \)?

\[
f_{X+Y}(\alpha) = \int_{-\infty}^{\infty} f_X(k) f_Y(\alpha - k) dk
\]

\[
f_X(k) f_Y(\alpha - k) = 1:
0 \leq \alpha \leq 2
0 \leq k \leq 1
0 \leq \alpha - k \leq 1
\alpha - 1 \leq k \leq \alpha
\]

The precise integration bounds on \( k \) depend on \( \alpha \).

What are the bounds on \( k \) when:

1. \( \alpha = 1/2 \)?
   \[
   0 \leq k \leq \alpha
   \int_{k=0}^{\alpha} 1 dk = \alpha = 1/2
   \]

2. \( \alpha = 3/2 \)?
   \[
   \alpha - 1 \leq k \leq 1
   \int_{k=\alpha-1}^{1} 1 dk = \alpha - 1 = 1/2
   \]

3. \( \alpha = 1 \)?
   \[
   0 \leq k \leq \alpha
   \int_{k=0}^{\alpha} 1 dk = \alpha = 1
   \]
   (the other bound works too)
Sum of independent Uniforms

Let $X \sim \text{Unif}(0,1)$ and $Y \sim \text{Unif}(0,1)$ be independent random variables. What is the distribution of $X + Y$, $f_{X+Y}$?

$$f_{X+Y}(\alpha) = \int_{-\infty}^{\infty} f_X(k) f_Y(\alpha - k) dk$$

$f_X(k)f_Y(\alpha - k) = 1$ when:

- $0 \leq \alpha \leq 2$
- $0 \leq k \leq 1$
- $0 \leq \alpha - k \leq 1$
- $\alpha - 1 \leq k \leq \alpha$

The precise integration bounds on $k$ depend on $\alpha$. 

$$f_{X+Y}(\alpha) = \begin{cases} 
\alpha & 0 \leq \alpha \leq 1 \\
2 - \alpha & 1 \leq \alpha \leq 2 \\
0 & \text{otherwise}
\end{cases}$$
Today’s plan

Independent RVs

Sum of independent RVs
- ✅ Binomial
- Convolution
- ✅ Poisson
- ✅ Normal
- ⚠️ Uniform

Expectation of sum of RVs (next class)
Properties of Expectation, extended to two RVs

1. Linearity:
\[ E[aX + bY + c] = aE[X] + bE[Y] + c \]

2. Expectation of a sum = sum of expectation:
\[ E[X + Y] = E[X] + E[Y] \]

3. Unconscious statistician:
\[ E[g(X,Y)] = \sum_{x} \sum_{y} g(x,y)p_{X,Y}(x,y) \]
\[ E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y)f_{X,Y}(x,y) \, dx \, dy \]

(we’ve seen this; we’ll prove this next)
Proof of expectation of a sum of RVs

\[ E[X + Y] = E[g(X, Y)] = \sum_x \sum_y g(x, y)p_{X,Y}(x, y) = \sum_x \sum_y (x + y)p_{X,Y}(x, y) \]

LOTUS, \( g(X, Y) = X + Y \)

\[ = \sum_x \sum_y xp_{X,Y}(x, y) + \sum_x \sum_y yp_{X,Y}(x, y) \]

Linearity of summations
(continuation: linearity of integrals)

\[ = \sum_x x \sum_y p_{X,Y}(x, y) + \sum_y y \sum_x p_{X,Y}(x, y) \]

Marginal PMFs for \( X \) and \( Y \)

\[ = \sum_x xp_X(x) + \sum_y yp_Y(y) \]

\[ = E[X] + E[Y] \]

Even if the joint distribution is unknown, you can calculate the expectation of sum as sum of expectations.

Example: \( E[\sum_{i=1}^n X_i] = \sum_{i=1}^n E[X_i] \) despite dependent trials \( X_i \)
Expectations of common RVs

\( X \sim \text{Bin}(n, p) \quad E[X] = np \)

\[
X = \sum_{i=1}^{n} X_i \quad \text{Let } X_i = \text{ith trial is heads} \\
X_i \sim \text{Ber}(p), E[X_i] = p
\]

\[
E[X] = E\left[ \sum_{i=1}^{n} X_i \right] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} p = np
\]
Expectations of common RVs

\[ X \sim \text{Bin}(n, p) \quad E[X] = np \]

\[ X = \sum_{i=1}^{n} X_i \]
Let \( X_i = \) \( i \)th trial is heads
\( X_i \sim \text{Ber}(p) \), \( E[X_i] = p \)

\[ E[X] = E \left[ \sum_{i=1}^{n} X_i \right] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} p = np \]

\[ Y \sim \text{NegBin}(r, p) \quad E[Y] = \frac{r}{p} \]

Suppose:

\[ Y = \sum_{i=1}^{?} Y_i \]

How should we define \( Y_i \)?

A. \( Y_i = \) \( i \)th trial is heads. \( Y_i \sim \text{Ber}(p) \), \( i = 1, \ldots, n \)

B. \( Y_i = \# \) trials to get \( i \)th success (after \((i - 1)\)th success)
\( Y_i \sim \text{Geo}(p) \), \( i = 1, \ldots, r \)

C. \( Y_i = \# \) successes in \( n \) trials
\( Y_i \sim \text{Bin}(n, p) \), \( i = 1, \ldots, r \), we look for \( P(Y_i = 1) \)
Expectations of common RVs

\[ X \sim \text{Bin}(n, p) \quad E[X] = np \]

\[ X = \sum_{i=1}^{n} X_i \quad \text{Let } X_i = \text{ith trial is heads} \]
\[ X_i \sim \text{Ber}(p), E[X_i] = p \]

\[ E[X] = E \left[ \sum_{i=1}^{n} X_i \right] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} p = np \]

\[ Y \sim \text{NegBin}(r, p) \quad E[Y] = \frac{r}{p} \]

\[ Y = \sum_{i=1}^{r} Y_i \quad \text{Let } Y_i = \text{# trials to get ith success (after } (i-1)\text{th success}) \]
\[ Y_i \sim \text{Geo}(p), E[Y_i] = \frac{1}{p} \]

\[ E[Y] = E \left[ \sum_{i=1}^{r} Y_i \right] = \sum_{i=1}^{r} E[Y_i] = \sum_{i=1}^{r} \frac{1}{p} = \frac{r}{p} \]