CS109 Lecture 1 Worksheet: ANSWER KEY

Foundations of Probability

Summer 2026

Lecture 1 Worksheet: Answer Key

Instructor / TA copy. Full worked solutions follow.

Problem 1: Sample Spaces and Events

(a) Each flip is H or T, and order matters, so

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}, \qquad |S| = 2^3 = 8.$$

(b) "At least two heads" means two or three heads:

$$E = \{HHH, HHT, HTH, THH\}, \qquad |E| = 4.$$

(c) "First flip is tails":

$$F = \{THH, THT, TTH, TTT\}, \qquad |F| = 4.$$

(d) They are **not** mutually exclusive. The outcome THH has a tails first *and* contains two heads, so

$$E \cap F = \{THH\} \neq \varnothing.$$

Mutually exclusive would require $E \cap F = \varnothing$.

Problem 2: Two Dice and the Right Sample Space

(a) With distinguishable dice, each die independently shows 1 through 6, so

$$|S| = 6 \times 6 = 36 \text{ equally likely ordered pairs.}$$

(b) The ordered pairs summing to 7 are $(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)$, so $|E| = 6$ and

$$P(E) = \frac{6}{36} = \frac{1}{6} \approx 0.167.$$

(c) Only $(1,1)$ gives a sum of 2, so

$$P(G) = \frac{1}{36} \approx 0.028.$$

(d) The 11 possible sums $\{2, 3, \ldots, 12\}$ are **not equally likely**. The rule $P(E) = |E|/|S|$ only applies when every outcome in the sample space has the same probability. A sum of 7 arises from 6 of the 36 equally likely ordered pairs, while a sum of 2 arises from just 1. Treating the 11 sums as equally likely silently assigns each probability $1/11$, which contradicts the fact (verifiable by enumerating the 36 pairs, or by simulation) that $P(\text{sum} = 7) = 6/36$ while $P(\text{sum} = 2) = 1/36$. The fix: build the sample space from the 36 equally likely *outcomes*, not the 11 unequally likely *sums*.

Problem 3: Drawing a Card, and When You May Add Probabilities

There are 52 equally likely cards.

(a) 13 hearts: $P(H) = \dfrac{13}{52} = \dfrac{1}{4}$.

(b) 4 aces: $P(A) = \dfrac{4}{52} = \dfrac{1}{13}$.

(c) H and A are **not** mutually exclusive: the ace of hearts is in both, so $H \cap A \neq \varnothing$. Therefore Axiom 3 does **not** apply and you may not simply add. Counting the union directly: $H \cup A$ contains the 13 hearts plus the 3 non-heart aces, so $|H \cup A| = 16$ and

$$P(H \cup A) = \frac{16}{52} = \frac{4}{13} \approx 0.308.$$

(Note that naive addition gives $\frac{13}{52} + \frac{4}{52} = \frac{17}{52}$, which double-counts the ace of hearts.)

(d) A (aces) and F (J, Q, K) share no cards, so they **are** mutually exclusive. By Axiom 3,

$$P(A \cup F) = P(A) + P(F) = \frac{4}{52} + \frac{12}{52} = \frac{16}{52} = \frac{4}{13} \approx 0.308.$$

(e) There are 13 spades, so $P(\text{spade}) = 13/52 = 1/4$. By the complement rule,

$$P(\text{not a spade}) = 1 - \frac{1}{4} = \frac{3}{4} = 0.75.$$

Problem 4: The Complement Is Your Friend

The sample space is $\{1, 2, \ldots, 12\}$, all equally likely, $|S| = 12$.

(a) Multiples of 3: $\{3, 6, 9, 12\}$, so $P(T) = \dfrac{4}{12} = \dfrac{1}{3}$. Values ≤ 4: $\{1, 2, 3, 4\}$, so $P(L) = \dfrac{4}{12} = \dfrac{1}{3}$.

(b) They are **not** mutually exclusive: 3 is both a multiple of 3 and ≤ 4, so $T \cap L = \{3\}$. The union is $\{1, 2, 3, 4, 6, 9, 12\}$, which has 7 elements:

$$P(T \cup L) = \frac{7}{12} \approx 0.583.$$

Naive addition gives $P(T) + P(L) = \frac{4}{12} + \frac{4}{12} = \frac{8}{12}$, which is too large by exactly $\frac{1}{12}$ because the shared outcome 3 was counted twice. (This previews inclusion-exclusion in a later lecture.)

(c) The event "not 7" is the complement of $\{7\}$, and $P(\{7\}) = 1/12$, so

$$P(\text{not } 7) = 1 - \frac{1}{12} = \frac{11}{12} \approx 0.917,$$

which lies in $[0, 1]$ as Axiom 1 requires.

Problem 5: What Probability Means, and a Check on the Axioms

(a) $P(\text{sunny}) \approx \dfrac{219}{365} = 0.6$. Interpretation: over many days drawn from the same conditions, we expect the long-run fraction of sunny days to be about 0.6; probability is the limiting relative frequency of the event as the number of trials grows.

(b) i. **Not possible.** Axiom 1 requires $0 \leq P(E) \leq 1$, so 1.2 is invalid.

ii. The classmate is **wrong**. There is no axiom forbidding $P(E) + P(F) > 1$. Axiom 3 bounds the probability of a *union* of mutually exclusive events by 1, but two overlapping (or even just unrelated) events can each be large; e.g., $P(E) = P(F) = 0.7$ gives a sum of 1.4. Only when E and F are mutually exclusive must $P(E) + P(F) = P(E \cup F) \leq 1$.

iii. By the identity $P(E^C) = 1 - P(E)$, we get $P(E^C) = 1 - 0.4 = 0.6$. That is exactly its value (not just an upper bound), and it is derived from Axioms 2 and 3 applied to the mutually exclusive pair E and E^C whose union is S.

End of answer key.