CS109 Lecture 2 Worksheet: ANSWER KEY

Conditional Probability and Bayes

Summer 2026

Lecture 2 Worksheet: Answer Key

Instructor / TA copy. Full worked solutions follow.

Problem 1 (Review): Sample Spaces, Mutually Exclusive Events, Complement

$S = \{1, 2, 3, 4, 5, 6\}$, all equally likely.

(a) $A = \{2, 4, 6\}$ so $P(A) = \frac{3}{6} = \frac{1}{2}$. $B = \{4, 5, 6\}$ so $P(B) = \frac{3}{6} = \frac{1}{2}$.

(b) $A \cap B = \{4, 6\} \neq \varnothing$, so they are **not** mutually exclusive.

(c) $A \cup B = \{2, 4, 5, 6\}$, so $P(A \cup B) = \frac{4}{6} = \frac{2}{3}$.

(d) $P(\text{not } 6) = 1 - P(6) = 1 - \frac{1}{6} = \frac{5}{6}$.

Problem 2: Conditioning Changes the Sample Space

There are 36 equally likely ordered pairs.

(a) Sum = 8: $(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)$, so $P(\text{sum} = 8) = \frac{5}{36} \approx 0.139$.

(b) Conditioning on "first die = 5" restricts us to the 6 outcomes $(5, 1), \ldots, (5, 6)$. Among these, only $(5, 3)$ gives a sum of 8, so

$$P(\text{sum} = 8 \mid D_1 = 5) = \frac{1}{6} \approx 0.167.$$

(c) Let F = "at least one die shows 5." There are 11 such outcomes (the five $(5, \cdot)$, the five $(\cdot, 5)$, minus the double-counted $(5, 5)$). The outcomes with sum 8 *and* a 5 are $(5, 3)$ and $(3, 5)$, so $|E \cap F| = 2$:

$$P(\text{sum} = 8 \mid \text{at least one } 5) = \frac{P(E \cap F)}{P(F)} = \frac{2/36}{11/36} = \frac{2}{11} \approx 0.182.$$

This differs from (b) because "at least one die is 5" is a larger, different conditioning event than "the first die is 5"; it admits the outcome $(3, 5)$ as well.

Problem 3: The Chain Rule (Drawing Without Replacement)

(a) $P(\text{both aces}) = P(A_1)\, P(A_2 \mid A_1) = \dfrac{4}{52} \cdot \dfrac{3}{51} = \dfrac{12}{2652} = \dfrac{1}{221} \approx 0.0045.$

(b) $P(\text{K then Q}) = \dfrac{4}{52} \cdot \dfrac{4}{51} = \dfrac{16}{2652} = \dfrac{4}{663} \approx 0.0060.$

(c) $P(\text{first three all hearts}) = \dfrac{13}{52} \cdot \dfrac{12}{51} \cdot \dfrac{11}{50} = \dfrac{1716}{132600} = \dfrac{11}{850} \approx 0.0129.$

Problem 4: Law of Total Probability (Spam Filter)

Let S = spam, S^C = ham, W = "contains the word free." Given $P(S) = 0.3$, $P(S^C) = 0.7$, $P(W \mid S) = 0.6$, $P(W \mid S^C) = 0.1$.

(a) By the law of total probability,

$$P(W) = P(W \mid S)P(S) + P(W \mid S^C)P(S^C) = (0.6)(0.3) + (0.1)(0.7) = 0.18 + 0.07 = 0.25.$$

(b) Tree: root splits into S (0.3) and S^C (0.7); each splits into W and W^C. The labeled branches are $S \to W$: 0.6, $S \to W^C$: 0.4, $S^C \to W$: 0.1, $S^C \to W^C$: 0.9. The two paths leading to W are $0.3 \times 0.6 = 0.18$ and $0.7 \times 0.1 = 0.07$, summing to 0.25.

Problem 5: Bayes' Theorem (Spam Filter, continued)

(a)

$$P(S \mid W) = \frac{P(W \mid S)P(S)}{P(W)} = \frac{(0.6)(0.3)}{0.25} = \frac{0.18}{0.25} = 0.72.$$

(b) **Prior** $P(S) = 0.3$ (belief before seeing the word); **likelihood** $P(W \mid S) = 0.6$ (how probable "free" is if the message is spam); **posterior** $P(S \mid W) = 0.72$ (updated belief after seeing "free").

(c) Seeing the word "free" raised our belief that the message is spam from 0.30 to 0.72, because "free" is much more likely in spam than in ham.

Problem 6: Bayes' Theorem and the Base-Rate Effect

Let D = has disease, $+$ = tests positive. Given $P(D) = 0.01$, $P(+ \mid D) = 0.98$, $P(+ \mid D^C) = 0.05$.

(a)

$$P(+) = P(+ \mid D)P(D) + P(+ \mid D^C)P(D^C) = (0.98)(0.01) + (0.05)(0.99) = 0.0098 + 0.0495 = 0.0593.$$

(b)

$$P(D \mid +) = \frac{P(+ \mid D)P(D)}{P(+)} = \frac{0.0098}{0.0593} \approx 0.165.$$

(c) **Natural frequencies (1000 people):** about 10 have the disease, and $10 \times 0.98 \approx 9.8$ of them test positive. Of the 990 healthy people, $990 \times 0.05 = 49.5$ test positive. Total positives $\approx 9.8 + 49.5 = 59.3$, of which only 9.8 truly have the disease: $9.8/59.3 \approx 0.165$. Intuition: the disease is rare (1%), so even a small false-positive rate applied to the large healthy majority produces more false positives than true positives.

Challenge Problem: Monty Hall

(a) Condition on the door you initially picked. With probability $\frac{1}{3}$ your first pick was correct (switching then loses), and with probability $\frac{2}{3}$ your first pick was wrong (the host is forced to reveal the only other empty door, so the remaining door holds the prize and switching wins). Therefore

$$P(\text{win} \mid \text{always switch}) = \frac{2}{3}.$$

(b) $P(\text{win} \mid \text{never switch}) = \frac{1}{3}$, exactly the chance your original pick was right.

(c) The host never opens the prize door and never opens your door; this deliberate choice concentrates the remaining $\frac{2}{3}$ probability onto the single unopened door you can switch to.