CS109 Lecture 4 Worksheet: ANSWER KEY

Counting

Summer 2026

Lecture 4 Worksheet: Answer Key

Instructor / TA copy. Full worked solutions follow.

Problem 1 (Review): Independence and "At Least One"

Each of 3 independent components is up with probability 0.95.

(a) $P(\text{all up}) = 0.95^3 = 0.857375 \approx 0.857$.

(b) $P(\text{at least one down}) = 1 - 0.95^3 = 1 - 0.857375 = 0.142625 \approx 0.143$.

Problem 2: Counting with Steps (the Product Rule)

(a) Each of the 4 positions independently has 10 choices: $10^4 = 10{,}000$ codes.

(b) No repeats: $10 \times 9 \times 8 \times 7 = 5040$ codes.

(c) Ordering 6 distinct known digits: $6! = 720$ possible orderings. (This is why distinct smudges still leave many possibilities.)

Problem 3: Fantasy Sports Draft (Spring 2026 Midterm Question)

(a) We count directly by breaking the construction into independent steps.

Step 1 – Choose which 2 of the 6 ranked slots are filled by goalkeepers (the remaining 4 slots will be filled by non-goalkeepers):

$$\binom{6}{2} = 15.$$

Step 2 – Fill the 2 goalkeeper slots by choosing and ordering 2 of the 5 available goalkeepers:

$$P(5,2) = 5 \cdot 4 = 20.$$

Step 3 – Fill the remaining 4 slots by choosing and ordering 4 of the 15 non-goalkeepers (8 forwards + 7 midfielders):

$$P(15,4) = 15 \cdot 14 \cdot 13 \cdot 12.$$

By the multiplication rule, the total number of such sequences is

$$\binom{6}{2} \cdot P(5,2) \cdot P(15,4) = 15 \cdot 20 \cdot (15 \cdot 14 \cdot 13 \cdot 12).$$

Alternate solution: First, we can count all the unordered groups of players with exactly 2 goalkeepers. There are $\binom{5}{2}$ ways to pick 2 goalkeepers, and $\binom{15}{4}$ ways to pick the remaining players. Using the step rule of counting, there are then $\binom{5}{2} \cdot \binom{15}{4}$ different unordered groups of players with 2 goalkeepers.

Then, we can count the number of ways to order these groups by multiplying by 6!. (That is, for each group, there are 6 choices for the first ranked player, 5 choices for the second, and so on.) This leaves us with $N = \binom{5}{2} \cdot \binom{15}{4} \cdot 6!$, which is equivalent to the above solution.

(b) `Given that exactly 4 of the 6 drafted players are forwards, we treat the 3 starters as chosen unifo`

The number of ways to choose 3 starters who are all forwards (from the 4 forwards in the draft) is $\binom{4}{3} = 4$.

Therefore,

$$P(\text{all 3 starters are forwards} \mid \text{exactly 4 forwards drafted}) = \frac{\binom{4}{3}}{\binom{6}{3}} = \frac{4}{20} = \frac{1}{5}.$$

Alternate solution: We can model this as the probability that the first player is a forward *and* the second player is a forward *and* the third player is a forward. This leads us to $\frac{4}{6} \cdot \frac{3}{5} \cdot \frac{2}{4} = \frac{1}{5}$.

Problem 4: Permutations of Letters (Semi-Distinct Objects)

(a) **BANANA** has 6 letters: three A's, two N's, one B:

$$\frac{6!}{3!\,2!\,1!} = \frac{720}{6 \cdot 2 \cdot 1} = 60.$$

(b) **MISSISSIPPI** has 11 letters: one M, four I's, four S's, two P's:

$$\frac{11!}{1!\,4!\,4!\,2!} = \frac{39{,}916{,}800}{1 \cdot 24 \cdot 24 \cdot 2} = 34{,}650.$$

(c) For n objects with repeated groups of sizes $n_1, n_2, \ldots, n_k$ (where $n_1 + \cdots + n_k = n$), the number of distinct arrangements is

$$\frac{n!}{n_1!\,n_2!\cdots n_k!}.$$

Problem 5: Combinations

(a) $\binom{52}{5} = \frac{52!}{5!\,47!} = 2{,}598{,}960$ hands.

(b) $\binom{10}{3} = \frac{10 \cdot 9 \cdot 8}{3!} = 120$ committees.

(c) Choosing which 3 of the 10 positions hold a 1 determines the string; order among the chosen positions does not matter, so the count is $\binom{10}{3} = 120$. It is a combination because we are selecting a subset of positions, not arranging distinguishable items.

Problem 6: Counting to Find Probabilities (Poker Hands)

Every hand is one of $\binom{52}{5} = 2{,}598{,}960$ equally likely outcomes.

(a) **Flush:** choose 1 of 4 suits, then 5 of that suit's 13 cards: $4 \cdot \binom{13}{5} = 4 \cdot 1287 = 5148$. (This count includes straight flushes.)

$$P(\text{flush}) = \frac{5148}{2{,}598{,}960} \approx 0.00198.$$

(b) **Four of a kind:** choose the rank (13 ways), take all four of its cards, then choose any 1 of the remaining 48 cards: $13 \cdot 48 = 624$.

$$P(\text{four of a kind}) = \frac{624}{2{,}598{,}960} \approx 0.000240.$$

Problem 7: Counting and Coin Flips

(a) Each flip has 2 outcomes and there are 10 flips: $2^{10} = 1024$ equally likely sequences.

(b) The number of sequences with exactly 4 heads is the number of ways to choose which 4 of the 10 positions are heads, $\binom{10}{4} = 210$. Hence

$$P(\text{exactly 4 heads}) = \frac{\binom{10}{4}}{2^{10}} = \frac{210}{1024} \approx 0.205.$$

(c) "At least 8 heads" means 8, 9, or 10 heads:

$$P = \frac{\binom{10}{8} + \binom{10}{9} + \binom{10}{10}}{2^{10}} = \frac{45 + 10 + 1}{1024} = \frac{56}{1024} \approx 0.0547.$$

Challenge Problem: Exactly Two Aces

(a) Choose 2 of the 4 aces and 3 of the 48 non-aces:

$$\binom{4}{2}\binom{48}{3} = 6 \cdot 17{,}296 = 103{,}776 \text{ hands.}$$

(b)

$$P(\text{exactly two aces}) = \frac{103{,}776}{2{,}598{,}960} \approx 0.0399.$$