CS109 Lecture 5 Worksheet: ANSWER KEY

The Binomial

Summer 2026

Lecture 5 Worksheet: Answer Key

Instructor / TA copy. Full worked solutions follow.

Problem 1 (Review): Counting and Probability

12 students: 7 juniors, 5 seniors; choose a committee of 4.

(a) Order does not matter, so the number of committees is

$$\binom{12}{4} = 495.$$

(b) Favorable: choose 2 of 5 seniors and 2 of 7 juniors:

$$P(\text{exactly 2 seniors}) = \frac{\binom{5}{2}\binom{7}{2}}{\binom{12}{4}} = \frac{10 \cdot 21}{495} = \frac{210}{495} \approx 0.424.$$

(c) All 4 from the 7 juniors:

$$P(\text{all juniors}) = \frac{\binom{7}{4}}{\binom{12}{4}} = \frac{35}{495} \approx 0.0707.$$

Problem 2: Random Variables and the PMF

Y = number of heads in 3 fair flips.

(a) The 8 equally likely outcomes are TTT; HTT, THT, TTH; HHT, HTH, THH; HHH. Counting heads:

$$P(Y = 0) = \tfrac{1}{8}, \quad P(Y = 1) = \tfrac{3}{8}, \quad P(Y = 2) = \tfrac{3}{8}, \quad P(Y = 3) = \tfrac{1}{8}.$$

(Each count is $\binom{3}{k}$ out of $2^3 = 8$, a preview of the binomial.)

(b) $\frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = 1$. This must always hold because the random variable takes on *exactly one* of its possible values on each trial, so the probabilities of the mutually exclusive, exhaustive outcomes sum to 1.

(c) $P(Y \geq 2) = P(Y = 2) + P(Y = 3) = \frac{3}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}$.

Problem 3: Using a PMF Someone Gives You

$P(X = k) = c \cdot 2^{-k}$ for $k = 1, 2, 3, \ldots$

(a) Require $\sum_{k=1}^{\infty} c\, 2^{-k} = c \cdot 1 = 1$, so $c = 1$. Thus $P(X = k) = 2^{-k}$.

(b) $P(X = 3) = 2^{-3} = \frac{1}{8} = 0.125$.

(c) $P(X \leq 2) = P(X = 1) + P(X = 2) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} = 0.75$.

Problem 4: Recognizing and Setting Up a Binomial

(a) **Binomial.** 8 independent trials, each a "success" (a 1) with probability $p = \frac{1}{2}$: $X \sim \text{Bin}(8, 0.5)$.

(b) **Binomial.** 1000 independent views, each clicked with probability $p = 0.01$: $X \sim \text{Bin}(1000, 0.01)$.

(c) **Not binomial.** Dealing without replacement means the trials are **not independent** and p (the chance the next card is a heart) changes as cards are removed. (This is a hypergeometric situation, counted with combinations instead.)

(d) For $X \sim \text{Bin}(8, 0.5)$,

$$P(X = 3) = \binom{8}{3} \left(\tfrac{1}{2}\right)^3 \left(\tfrac{1}{2}\right)^5 = \binom{8}{3} \left(\tfrac{1}{2}\right)^8 = \frac{56}{256} \approx 0.219.$$

Problem 5: Server Redundancy (Cumulative Binomial)

$X \sim \text{Bin}(7, 0.8)$, number of servers alive.

(a)

$$P(X = k) = \binom{7}{k} (0.8)^k (0.2)^{7-k}, \qquad k = 0, 1, \ldots, 7.$$

(b) Fewer than 2 alive means $k = 0$ or $k = 1$:

$$P(X = 0) = (0.2)^7 = 0.0000128,$$

$$P(X = 1) = \binom{7}{1} (0.8)(0.2)^6 = 7 \cdot 0.8 \cdot 0.000064 = 0.0003584.$$

$$P(X < 2) = 0.0000128 + 0.0003584 = 0.0003712 \approx 0.00037.$$

(c) $P(X \geq 2) = 1 - P(X < 2) = 1 - 0.0003712 \approx 0.99963$. The redundant design is very reliable.

Problem 6: Wisdom of the Crowds (adapted from the Fall 2017 Midterm)

There are 20 experts, each voting Correct independently with probability 0.7. Let X be the number who vote Correct.

(a) X is a **binomial** random variable: $X \sim \text{Bin}(20, 0.7)$, with

$$P(X = k) = \binom{20}{k} (0.7)^k (0.3)^{20-k}, \qquad k = 0, 1, \ldots, 20.$$

(b)

$$P(X = 14) = \binom{20}{14} (0.7)^{14} (0.3)^6 = 38760 \cdot (0.7)^{14} (0.3)^6 \approx 0.192.$$

(c) "At least 18" means $k = 18, 19, 20$:

$$P(X \geq 18) = \sum_{k=18}^{20} \binom{20}{k} (0.7)^k (0.3)^{20-k} \approx 0.0355.$$

Note: the original midterm continued by asking for the chance the Correct answer reaches at least 101 of all 200 votes, which combines the experts with 180 random non-experts and is handled later in the course with the normal approximation. Here we stay within the binomial.

Problem 7: Winning a Best-of-Series

$X \sim \text{Bin}(7, 0.55)$, number of games the Warriors win.

(a) Win the series $\iff X \geq 4$:

$$P(X \geq 4) = \sum_{k=4}^{7} \binom{7}{k} (0.55)^k (0.45)^{7-k} \approx 0.608.$$

(b) The proposed shortcut is wrong for two related reasons. First, the events "these particular 4 games are wins" for different choices of 4 slots are **not mutually exclusive** (a 5-win outcome is counted under several of the $\binom{7}{4}$ choices), so you cannot just multiply by $\binom{7}{4}$ and add. Second, dropping the (0.45) factors fails a sanity check: at $p = 1$ the expression gives $\binom{7}{4} = 35 \neq 1$, but a sure win must have probability 1. The correct approach sums the **exact** binomial PMF over $k = 4, 5, 6, 7$, where each term already fixes the remaining games as losses.

Challenge Problem: The Galton Board

(a) Each of the 5 levels is an independent trial with two outcomes (left/right), and "right" has the same probability 0.5 every time. The number of rights is a count of successes in $n = 5$ independent, equal-probability trials, so $B \sim \text{Bin}(5, 0.5)$.

(b) $P(B = k) = \binom{5}{k} \left(\frac{1}{2}\right)^5 = \frac{\binom{5}{k}}{32}$:

$$P(0) = \tfrac{1}{32},\ P(1) = \tfrac{5}{32},\ P(2) = \tfrac{10}{32},\ P(3) = \tfrac{10}{32},\ P(4) = \tfrac{5}{32},\ P(5) = \tfrac{1}{32}.$$

These sum to $32/32 = 1$, as required.

(c) Buckets 2 and 3 are most likely (each $\frac{10}{32} \approx 0.3125$). The shape is symmetric because $p = 0.5$ makes $\binom{5}{k}$ symmetric in k, and the binomial coefficients $1, 5, 10, 10, 5, 1$ rise to the middle and fall off, giving the bell-like profile. Dropping many marbles makes the observed histogram approach this PMF.