CS109 Lecture 5 Worksheet: The Binomial

Random Variables, Probability Mass Functions, and the Binomial Distribution

Summer 2026

Lecture 5 Worksheet: The Binomial

Topics: random variables, probability mass functions (PMFs) and their properties, recognizing a binomial scenario, the binomial PMF $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$, and cumulative ("at least k" / "fewer than k") binomial probabilities.

How to use this sheet. Work in small groups. A TA and the instructor will circulate to help. We will go over selected solutions on the board. You do not need to finish every part; aim for understanding over speed. You may leave answers as expressions with $\binom{n}{k}$, powers, and factorials unless a decimal is requested, and a quick `scipy.stats.binom` check is suggested where useful.

Problem 1 (Review of Lecture 4): Counting and Probability

A class has 12 students: 7 juniors and 5 seniors. A committee of 4 students is chosen uniformly at random (order does not matter).

(a) How many possible committees are there?

(b) What is the probability that the committee has **exactly 2 seniors**? Count the favorable committees as (ways to choose 2 of the 5 seniors) $\times$ (ways to choose 2 of the 7 juniors).

(c) What is the probability that the committee is **all juniors**?

Problem 2: Random Variables and the PMF

Let Y be the number of heads in **3 independent flips** of a fair coin.

(a) List the 8 equally likely outcomes (e.g. HHT) and use them to find $P(Y = 0)$, $P(Y = 1)$, $P(Y = 2)$, and $P(Y = 3)$. This table of values is the **probability mass function (PMF)** of Y.

(b) Verify that $\sum_k P(Y = k) = 1$. Why must this always hold for any random variable?

(c) Find $P(Y \geq 2)$ (at least 2 heads).

Problem 3: Using a PMF Someone Gives You

Let X be the number of earthquakes in California next year. A seismologist hands you the PMF

$$P(X = k) = c \cdot \frac{1}{2^k}, \qquad k = 1, 2, 3, \ldots$$

(a) A valid PMF must sum to 1. Using $\sum_{k=1}^{\infty} \frac{1}{2^k} = 1$, find the constant c.

(b) Find $P(X = 3)$.

(c) Find $P(X \leq 2)$.

Problem 4: Recognizing and Setting Up a Binomial

For each scenario below, state whether it is a binomial. If it is, give the parameters n and p and write $X \sim \text{Bin}(n, p)$. If it is not, say what assumption fails.

(a) The number of 1's in a randomly generated bit string of length 8, where each bit is independently 1 with probability $\frac{1}{2}$.

(b) The number of clicks when an ad is shown 1000 times and each view is independently clicked with probability 0.01.

(c) The number of cards that are hearts when 5 cards are dealt **without replacement** from a 52-card deck.

(d) For the bit-string scenario in part (a), write an expression for the probability of **exactly three 1's**, then give a decimal (a quick check: it is about 0.219).

Problem 5: Server Redundancy (Cumulative Binomial)

A network stays functional as long as **at least 2 of its 7 servers** are alive. Each server is alive independently with probability 0.8. Let X be the number of servers alive, so $X \sim \text{Bin}(7, 0.8)$.

(a) Write the binomial PMF $P(X = k)$ for this X.

(b) Find $P(X < 2)$, the probability that **fewer than 2** servers are alive (the network fails). Compute it as $P(X = 0) + P(X = 1)$.

(c) Use part (b) to find the probability the network stays functional, $P(X \geq 2)$.

Problem 6: Wisdom of the Crowds (adapted from the Fall 2017 Midterm)

In *Who Wants to be a Millionaire*, a contestant may "ask the audience." There are 200 audience members, each choosing between a Correct and an Incorrect answer. Assume 10% of the audience are **experts**, so there are 20 experts, and each expert independently votes for the Correct answer with probability 0.7.

Let X be the number of experts who vote for the Correct answer.

(a) What kind of random variable is X? Give its parameters and write down $P(X = k)$ for k between 0 and 20.

(b) Write an expression for the probability that **exactly 14** of the experts vote Correct, then give a decimal (a quick check: about 0.192).

(c) Write an expression for the probability that **at least 18** of the experts vote Correct.

Problem 7: Winning a Best-of-Series

In a best-of-7 series, the Warriors play the Celtics. Each game is independent and the Warriors win each game with probability 0.55. The Warriors win the series if they win **at least 4 games** (assume all 7 games are played). Let X be the number of games the Warriors win, so $X \sim \text{Bin}(7, 0.55)$.

(a) Write an expression for the probability the Warriors win the series, $P(X \geq 4)$, as a sum of binomial terms.

(b) A classmate proposes computing $P(X \geq 4)$ as

$$\binom{7}{4}(0.55)^4 \text{ (anything for the rest)} = \binom{7}{4}(0.55)^4.$$

Explain in one or two sentences why this is **wrong**. (Hint: think about what $\binom{7}{4}(0.55)^4(0.45)^0$ would give if $p = 1$, and whether the chosen-slot events are mutually exclusive.)

Challenge Problem (optional): The Galton Board

A Galton board has 5 levels of pins. A marble starts at the top, and at each level it goes left or right with equal probability $\frac{1}{2}$, independently. Let B be the number of times the marble goes **right**, which determines the bucket (0 through 5) it lands in.

(a) Explain why $B \sim \text{Bin}(5, 0.5)$.

(b) Find the probability the marble lands in each of the 6 buckets. Express each as $\binom{5}{k}/2^5$.

(c) Which bucket(s) are most likely, and why does the histogram of many marbles look symmetric and bell-shaped?