CS109 Lecture 6 Worksheet: ANSWER KEY

Moments (Expectation)

Summer 2026

Lecture 6 Worksheet: Answer Key

Instructor / TA copy. Full worked solutions follow.

Problem 1 (Review): Binomial PMF

$X \sim \text{Bin}(5, 0.25)$.

(a) $P(X = k) = \binom{5}{k}(0.25)^k(0.75)^{5-k}, \quad k = 0, 1, \ldots, 5.$

(b) $P(X = 2) = \binom{5}{2}(0.25)^2(0.75)^3 = 10 \cdot 0.0625 \cdot 0.421875 \approx 0.264.$

(c) $P(X \geq 1) = 1 - P(X = 0) = 1 - (0.75)^5 = 1 - 0.2373 \approx 0.763.$

Problem 2: Naming the Classic Discrete Random Variables

(a) **Bernoulli**, $\text{Bern}(p = 0.01)$ (a single trial, success = click).

(b) **Geometric**, $\text{Geo}(p = 0.2)$ (trials until the first success).

(c) **Negative Binomial**, $\text{NegBin}(r = 3,\ p = 0.53)$ (trials until the 3rd success).

(d) **Binomial**, $\text{Bin}(n = 20,\ p = 0.5)$ (successes in a fixed number of trials).

Problem 3: Expectation from a PMF

(a) $E[X] = \sum_{x=1}^{6} x \cdot \frac{1}{6} = \frac{1+2+3+4+5+6}{6} = \frac{21}{6} = 3.5$. No, 3.5 is not a face of the die: an expectation need not be an attainable value.

(b) Pick a class uniformly: $E[X] = \frac{1}{3}(5) + \frac{1}{3}(10) + \frac{1}{3}(150) = \frac{165}{3} = 55.$

(c) Pick a student uniformly. A student is in the size-5 class with probability $\frac{5}{165}$, etc., so

$$E[X] = 5 \cdot \tfrac{5}{165} + 10 \cdot \tfrac{10}{165} + 150 \cdot \tfrac{150}{165} = \tfrac{25+100+22500}{165} = \tfrac{22625}{165} \approx 137.1.$$

It is larger because large classes contain *more students*, so a randomly chosen student is much more likely to sit in the size-150 class. (This is the classic "class-size paradox.")

Problem 4: Linearity and LOTUS

X = fair die, $E[X] = 3.5$.

(a) $E[W] = E[2X-1] = 2E[X] - 1 = 2(3.5) - 1 = 6$.

(b) $E[X^2] = \sum_{x=1}^{6} x^2 \cdot \frac{1}{6} = \frac{1+4+9+16+25+36}{6} = \frac{91}{6} \approx 15.17$.

(c) $(E[X])^2 = 3.5^2 = 12.25 \neq 15.17 = E[X^2]$. In general $E[X^2] \geq (E[X])^2$; the difference $E[X^2] - (E[X])^2 = 15.17 - 12.25 = 2.917$ is the variance (next lecture).

Problem 5: Expectations of the Classic Random Variables

(a) $Y \sim \text{Bin}(500, 0.53)$, so $E[Y] = np = 500 \cdot 0.53 = 265$ makes.

(b) At $p = 0.63$: $E[Y] = 500 \cdot 0.63 = 315$. That is $315 - 265 = 50$ more makes. (Equivalently $500 \cdot 0.10$, by linearity.)

(c) $X \sim \text{Geo}(0.2)$, so $E[X] = 1/p = 1/0.2 = 5$. Intuition: if each attempt succeeds a fraction p of the time, then on average it takes $1/p$ attempts to get one success.

Problem 6: Daycare Show-Ups (Spring 2019 Midterm)

$X \sim \text{Bin}(6, \frac{5}{6})$.

(a)

$$P(X=5) + P(X=6) = \binom{6}{5}\left(\tfrac{5}{6}\right)^5\left(\tfrac{1}{6}\right) + \left(\tfrac{5}{6}\right)^6 = \tfrac{3125}{7776} + \tfrac{15625}{46656} = \tfrac{34375}{46656} \approx 0.737.$$

(b) Revenue $= 50X$, so by linearity $E[50X] = 50\,E[X] = 50 \cdot np = 50 \cdot 6 \cdot \frac{5}{6} = 50 \cdot 5 = \$250$.

(c) Let a be the answer to (a). Staff cost is \$200 if 0–4 show (probability $1-a$) and \$400 if 5 or 6 show (probability a):

$$E[\text{cost}] = 200(1-a) + 400a = 200(1+a) = 200(1.737) \approx \$347.4.$$

Problem 7: Linearity Without Independence

(a) X_i takes only the values 0 and 1, so it is Bernoulli. The shuffled card in position i is equally likely to be any of the 52 cards, so it matches the "correct" card with probability $\frac{1}{52}$. Thus $E[X_i] = P(X_i = 1) = \frac{1}{52}$.

(b) By linearity (which holds even though the X_i are **not** independent),

$$E[X] = \sum_{i=1}^{52} E[X_i] = 52 \cdot \tfrac{1}{52} = 1.$$

On average exactly one card lands in its correct position.

Challenge Problem: Why $E[\textbf{Bin}(n,p)] = np$

(a) Each $Y_i \sim \text{Bern}(p)$, an indicator of success on trial i, with $E[Y_i] = p$.

(b) Since $X = \sum_{i=1}^{n} Y_i$, linearity gives

$$E[X] = \sum_{i=1}^{n} E[Y_i] = \sum_{i=1}^{n} p = np.$$

No binomial summation needed: writing the binomial as a sum of Bernoullis makes its expectation immediate.