CS109 Lecture 6 Worksheet: Moments (Expectation)

Classic Discrete Random Variables and Expected Value

Summer 2026

Lecture 6 Worksheet: Moments (Expectation)

Topics: the classic discrete random variables (Bernoulli, Geometric, Negative Binomial), expectation $E[X] = \sum_x x\, P(X = x)$, linearity of expectation, the Law of the Unconscious Statistician (LOTUS), and the expectations of the classic random variables.

How to use this sheet. Work in small groups. A TA and the instructor will circulate to help. We will go over selected solutions on the board. You do not need to finish every part; aim for understanding over speed. You may leave answers as exact fractions or expressions unless a decimal is requested.

Problem 1 (Review of Lecture 5): Binomial PMF

A quiz has 5 multiple-choice questions, each with 4 equally likely options. A student guesses uniformly at random on each. Let X be the number of correct answers, so $X \sim \text{Bin}(5, 0.25)$.

(a) Write the PMF $P(X = k)$.

(b) Find the probability of **exactly 2** correct, $P(X = 2)$.

(c) Find the probability of **at least 1** correct, $P(X \geq 1)$, using the complement.

Problem 2: Naming the Classic Discrete Random Variables

For each scenario, name the classic random variable (Bernoulli, Binomial, Geometric, or Negative Binomial) and give its parameter(s).

(a) Whether a single ad view results in a click (click probability 0.01).

(b) The number of people you date, each independently a "life partner" with probability 0.2, **until** you find your first life partner.

(c) The number of independent free throws Shaq attempts **until** he has made his 3rd basket (each made with probability 0.53).

(d) The number of heads in 20 independent fair coin flips.

Problem 3: Expectation from a PMF

(a) Let X be the result of one roll of a fair six-sided die. Compute $E[X] = \sum_{x=1}^{6} x\, P(X = x)$. Is $E[X]$ a value the die can actually show?

(b) A university has 3 classes of sizes 5, 10, and 150. You pick a class uniformly at random; let X be its size. Find $E[X]$.

(c) Now pick a **student** uniformly at random from all $5 + 10 + 150 = 165$ students, and let X be the size of **that student's** class. Find $E[X]$. Explain why this "average class size experienced by a student" is larger than the answer to (b).

Problem 4: Linearity and LOTUS

Let X be the result of one roll of a fair six-sided die ($E[X] = 3.5$).

(a) You win \$2 per pip minus a \$1 entry fee, so your winnings are $W = 2X - 1$. Use **linearity**, $E[aX + b] = a\,E[X] + b$, to find $E[W]$.

(b) Use **LOTUS**, $E[g(X)] = \sum_x g(x)P(X = x)$, to find $E[X^2]$.

(c) Is $E[X^2]$ equal to $(E[X])^2$? Compute both and comment. (This gap is exactly what *variance* will measure next lecture.)

Problem 5: Expectations of the Classic Random Variables

Recall (and use) the expectation formulas: $E[\text{Bern}(p)] = p$, $E[\text{Bin}(n, p)] = np$, $E[\text{Geo}(p)] = 1/p$.

(a) Shaq makes each free throw with probability 0.53. Over 500 attempts, let Y be the number he makes. What is $E[Y]$?

(b) If Shaq were 10 percentage points better (probability 0.63), how many more makes would you expect out of 500?

(c) Each person you date is a life partner with probability 0.2, independently. Let X be the number of people you date until (and including) your first life partner. What is $E[X]$, and what is the intuition behind the formula $1/p$?

Problem 6: Daycare Show-Ups (Spring 2019 Midterm)

A daycare has 6 enrolled babies. Each shows up on a given day independently with probability $\frac{5}{6}$. Let X be the number who show up.

(a) What is the probability that **either 5 or 6** babies show up?

(b) If you charge \$50 per baby who shows up, what is your **expected revenue**? (Hint: revenue is $50X$; use linearity and $E[\text{Bin}(n, p)] = np$.)

(c) If 0 to 4 babies show up you hire one staff member; if 5 or 6 show up you hire two. Each staff member costs \$200. Using your answer a from part (a), write the **expected staff cost**.

Problem 7: Linearity Without Independence

You shuffle a standard deck of 52 cards uniformly at random. For each position i, let $X_i = 1$ if the card in position i matches the card that "should" be there in sorted order (an exact match), and 0 otherwise. Let $X = \sum_{i=1}^{52} X_i$ be the total number of matches.

(a) Argue that each X_i is a Bernoulli random variable and find $E[X_i] = P(X_i = 1)$.

(b) Use linearity of expectation (which does **not** require independence) to find $E[X]$, the expected number of cards in their correct position.

Challenge Problem (optional): Why $E[\mathbf{Bin}(n, p)] = np$

Let $X \sim \text{Bin}(n, p)$. Write $X = Y_1 + Y_2 + \cdots + Y_n$, where $Y_i = 1$ if trial i is a success and 0 otherwise.

(a) What is the distribution of each Y_i, and what is $E[Y_i]$?

(b) Use linearity of expectation to show $E[X] = np$, without touching the binomial PMF or any hard sums.