CS109 Lecture 14 Worksheet: Beta

The Random Variable for Probabilities

Summer 2026

Lecture 14 Worksheet: Beta

Topics: Representing belief about an unknown probability as a random variable, the Beta distribution and its parameters, updating a uniform or Beta prior with observed successes/failures (conjugacy), Laplace smoothing, and using the Beta for decisions (expectation, probabilities via the CDF, Thompson sampling).

How to use this sheet. Work in small groups. A TA and the instructor will circulate to help. We will go over selected solutions on the board. You do not need to finish every part; aim for understanding over speed. Keep fractions exact where you can.

Problem 1 (Review of Lecture 13): Five Quick Rolls

A fair 6-sided die is rolled 5 times. What is the probability of getting exactly two 3s, two 5s, and one 6?

Problem 2: The Robot Gripper

A warehouse robot attempts to grasp objects. Before any data, your belief about X, its per-attempt success probability, is $\text{Uni}(0,1)$. You watch 8 attempts: 6 succeed and 2 fail.

(a) What is your posterior belief about X? Give the distribution and its parameters.

(b) What is $E[X]$? What is the mode of the posterior (the most likely value of X)? Why are they different from each other, and how does each compare to the frequentist estimate $6/8$?

(c) Sketch (roughly) the shape of the posterior PDF. Where is it centered, and is it symmetric?

Problem 3: Updating a Beta Prior

From past deployments, your belief about a delivery drone's success probability is $X \sim \text{Beta}(4,2)$. This week the drone makes 8 deliveries: 3 succeed and 5 fail.

(a) What is the posterior distribution of X? (Use conjugacy: just update the parameters.)

(b) Interpret the prior $\text{Beta}(4,2)$ in terms of "imagined" successes and failures.

(c) *Laplace smoothing* uses the prior $\text{Beta}(2,2)$ (one imagined success, one imagined failure). Under this prior, after observing s successes in n trials, what is the mode of the posterior? (This is the "Laplace-smoothed" probability estimate. Recall the mode of $\text{Beta}(a,b)$ is $\frac{a-1}{a+b-2}$.)

Problem 4: Puppy Training (problem from the Fall 2025 Midterm)

A new training method is tested on four puppies. The method successfully gets three of the puppies to sit when told, but one puppy ignores the command. Assume a Uniform$(0, 1)$ prior for the probability that the training method succeeds. What is the variance of the posterior distribution (after observing the four puppies) for this probability? Recall that for $X \sim \text{Beta}(a, b)$:

$$\text{Var}(X) = \frac{ab}{(a+b)^2(a+b+1)}.$$

Problem 5: Street Parking (problem from the Fall 2023 Final)

In order to make optimal parking decisions you need to represent your belief in X, the probability that parking spaces on a particular road will be open (parking spaces are either open or not). Before observations you believe the probability that a space is open is Uniform$(0, 1)$. You pass 10 parking spots on the road, 9 of which are full, one of which was open. What is your posterior belief in X if you consider the 10 spots to be IID samples? Also give $E[X]$.

Problem 6: Medicine with a Uniform Prior (pset4: warmup_beta)

You are developing medicine that sometimes has a desired effect, and sometimes does not. With FDA approval, you are allowed to test your medicine on 9 patients. You observe that 7 have the desired outcome. Your belief as to the probability of the medicine having the desired effect before running any experiments was uniform.

Compute a random variable representation for p, the probability the medicine has the desired effect. Calculate the probability that $p > 0.6$. (Set up the answer as a Beta CDF expression; in lecture we use `stats.beta.cdf` to evaluate it.)

Problem 7: Thompson Sampling (pset4: learning_while_helping)

Your clinic has two experimental drugs. So far, drug A has been given 8 times with 6 successes; drug B has been given 2 times with 1 success. Beliefs started uniform for both.

(a) Give the Beta distribution representing your current belief about each drug's success probability, and each belief's expected value.

(b) Describe how Thompson sampling chooses which drug to give the next patient.

(c) Drug A has the higher expected success probability under your current beliefs. Why might Thompson sampling still sometimes choose drug B, and why is that behavior desirable? (One or two sentences about exploration vs. exploitation.)

Challenge Problem (optional): Two Beta Facts

(a) Using the Beta PDF $f(x) = \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1}$ on $[0, 1]$, show that Beta$(1, 1)$ is exactly Uni$(0, 1)$.

(b) Your belief about a coin's heads probability is $X \sim \text{Beta}(a, b)$. Show that the probability the *next* flip is heads equals $E[X] = \frac{a}{a+b}$. (Hint: condition on $X = x$ and use the law of total probability, $P(\text{heads}) = \int_0^1 P(\text{heads} \mid X = x) f(x)\, dx$.)