CS109 Lecture 15 Worksheet: ANSWER KEY

Adding Random Variables & the CLT

Summer 2026

Lecture 15 Worksheet: Answer Key

Instructor / TA copy. Problem-set solutions are hidden in the standard build; run `./convert-to-pdf.sh 15 solutions` for the full instructor/TA edition.

Problem 1 (Review): Click-Through Belief

(a) Uniform prior is $\text{Beta}(1, 1)$; 9 successes and 3 failures give $X \sim \text{Beta}(10, 4)$.

(b) $E[X] = \frac{10}{14} = \frac{5}{7} \approx 0.714$.

Problem 2: Sums with Closed Forms

(a) $X + Y \sim \text{Bin}(40,\ 0.25)$. The argument treats all $12 + 28$ trials as one pooled sequence of independent trials with the *same* success probability; with different p's the pooled trials would not be identical, and the sum is not a binomial.

(b) Independent Poissons add: total $\sim \text{Poi}(2.2 + 3.8) = \text{Poi}(6)$.

$$P(\text{total} = 5) = \frac{e^{-6}\, 6^5}{5!} \approx 0.161.$$

(c) Independent normals add, means and variances both: $X + Y \sim N(30,\ 25)$, so $\sigma = 5$ and

$$P(X + Y > 40) = 1 - \Phi\left(\frac{40 - 30}{5}\right) = 1 - \Phi(2) \approx 0.0228.$$

Problem 3: Convolution by Hand

(a)

$$P(X + Y = n) = \sum_{k=0}^{n} P(X = k)\, P(Y = n - k).$$

(b)

$$P(X+Y = 2) = P(X = 1)P(Y = 1)+P(X = 2)P(Y = 0) = (0.3)(0.6)+(0.2)(0.4) = 0.18+0.08 = 0.26.$$

(The term $P(X = 0)P(Y = 2)$ contributes 0 since $Y \leq 1$.)

(c) The event $\{X + Y = n\}$ is the mutually exclusive OR over the ways to split n between X and Y, so probabilities add; independence lets each term factor as $P(X = k)P(Y = n - k)$.

Problem 4: ELO and Zero-Sum Games

(a) $D = A + (-B)$ is a sum of independent normals: $\mu = 1650 - 1500 = 150$ and $\text{Var} = 200^2 + (-1)^2\, 200^2 = 80{,}000$. So $D \sim N(150,\ 80{,}000)$, $\sigma = \sqrt{80{,}000} \approx 282.8$. (Variances *add* even for a difference.)

(b)
$$P(D > 0) = 1 - \Phi\left(\frac{0 - 150}{282.8}\right) = \Phi\left(\frac{150}{282.8}\right) = \Phi(0.53) \approx 0.70.$$

(c) Toward 0.5. The mean gap stays 150 but the spread of D grows, so the z-score of 0 shrinks toward 0; more randomness makes the upset more likely.

Problem 5: Truncation Error (Winter 2025 Final)

(a) Truncating to 3 decimals removes an amount uniform over one truncation bin: $(X_i - Y_i) \sim \text{Uni}(0,\ 0.001)$.

(b) $X - Y = \sum_{i=1}^{1000}(X_i - Y_i)$ is a sum of 1000 i.i.d. RVs, so by the CLT it is approximately normal with

$$\mu = 1000 \cdot \frac{0.001}{2} = 0.5, \qquad \sigma^2 = 1000 \cdot \frac{(0.001)^2}{12} = 8.33 \times 10^{-5}, \quad \sigma \approx 0.00913.$$

$$P(X - Y > 0.51) = 1 - \Phi\left(\frac{0.51 - 0.5}{0.00913}\right) = 1 - \Phi(1.10) \approx 0.137.$$

(The exam also accepted the two-tailed reading $P(|X - Y| > 0.51)$; the left tail $P(X - Y < -0.51)$ is essentially 0 here since the errors are all non-negative, so the answer is unchanged.)

Problem 6: Does the CLT Apply? (Fall 2023 Final)

No! The CLT requires the summed variables to be independent **and identically distributed**. These six are independent but come from six different distributions, so the CLT makes no claim about X. (With only $n = 6$ terms, "large n" would also be a stretch.)

Problem 7: Rolling Until 300 (pset5: sum_dice)

"At least 80 rolls are needed" happens exactly when the first 79 rolls have not yet passed 300: $S_{79} = \sum_{i=1}^{79} X_i \leq 300$. By the CLT,

$$S_{79} \approx N\left(\mu = 79 \cdot \tfrac{7}{2} = 276.5, \quad \sigma^2 = 79 \cdot \tfrac{35}{12} \approx 230.4\right), \qquad \sigma \approx 15.18.$$

With continuity correction (S_{79} is integer-valued, so $S_{79} \leq 300$ becomes ≤ 300.5):

$$P(S_{79} \leq 300) \approx \Phi\left(\frac{300.5 - 276.5}{15.18}\right) = \Phi(1.58) \approx 0.943.$$

(A simulation of 200{,}000 runs gives ≈ 0.943.)

Challenge Problem: Sum of 30 Betas (pset5: sum_betas)

Each $X_i \sim \text{Beta}(4, 2)$ has $E[X_i] = \frac{4}{6} = \frac{2}{3}$ and $\text{Var}(X_i) = \frac{4 \cdot 2}{6^2 \cdot 7} = \frac{2}{63}$. By the CLT,

$$X \approx N\big(\mu = 30 \cdot \tfrac{2}{3} = 20, \quad \sigma^2 = 30 \cdot \tfrac{2}{63} = \tfrac{60}{63} \approx 0.952\big), \qquad \sigma \approx 0.976.$$

$$P(19 < X < 20) \approx \Phi\left(\frac{20 - 20}{0.976}\right) - \Phi\left(\frac{19 - 20}{0.976}\right) = \Phi(0) - \Phi(-1.02) \approx 0.5 - 0.153 = 0.347.$$

No continuity correction because the X_i (and hence X) are continuous random variables — there are no integer gaps to correct for.