

CS109 Lecture 15 Worksheet: ANSWER KEY

Adding Random Variables & the CLT

Summer 2026

Lecture 15 Worksheet: Answer Key

Instructor / TA copy. Problem-set solutions are hidden in the standard build; run `./convert-to-pdf.sh 15 solutions` for the full instructor/TA edition.

Problem 1 (Review): Click-Through Belief

- (a) Uniform prior is $\text{Beta}(1, 1)$; 9 successes and 3 failures give $X \sim \text{Beta}(10, 4)$.
- (b) $E[X] = \frac{10}{14} = \frac{5}{7} \approx 0.714$.

Problem 2: Sums with Closed Forms

- (a) $X + Y \sim \text{Bin}(40, 0.25)$. The argument treats all $12 + 28$ trials as one pooled sequence of independent trials with the *same* success probability; with different p 's the pooled trials would not be identical, and the sum is not a binomial.
- (b) Independent Poissons add: $\text{total} \sim \text{Poi}(2.2 + 3.8) = \text{Poi}(6)$.

$$P(\text{total} = 5) = \frac{e^{-6} 6^5}{5!} \approx 0.161.$$

- (c) Independent normals add, means and variances both: $X + Y \sim N(30, 25)$, so $\sigma = 5$ and

$$P(X + Y > 40) = 1 - \Phi\left(\frac{40 - 30}{5}\right) = 1 - \Phi(2) \approx 0.0228.$$

Problem 3: Convolution by Hand

- (a)

$$P(X + Y = n) = \sum_{k=0}^n P(X = k) P(Y = n - k).$$

- (b)

$$P(X + Y = 2) = P(X = 1)P(Y = 1) + P(X = 2)P(Y = 0) = (0.3)(0.6) + (0.2)(0.4) = 0.18 + 0.08 = 0.26.$$

(The term $P(X = 0)P(Y = 2)$ contributes 0 since $Y \leq 1$.)

- (c) The event $\{X + Y = n\}$ is the mutually exclusive OR over the ways to split n between X and Y , so probabilities add; independence lets each term factor as $P(X = k)P(Y = n - k)$.

Problem 4: ELO and Zero-Sum Games

- (a) $D = A + (-B)$ is a sum of independent normals: $\mu = 1650 - 1500 = 150$ and $\text{Var} = 200^2 + (-1)^2 200^2 = 80,000$. So $D \sim N(150, 80,000)$, $\sigma = \sqrt{80,000} \approx 282.8$. (Variances *add* even for a difference.)

(b)

$$P(D > 0) = 1 - \Phi\left(\frac{0 - 150}{282.8}\right) = \Phi\left(\frac{150}{282.8}\right) = \Phi(0.53) \approx 0.70.$$

- (c) Toward 0.5. The mean gap stays 150 but the spread of D grows, so the z-score of 0 shrinks toward 0; more randomness makes the upset more likely.

Problem 5: Truncation Error (Winter 2025 Final)

- (a) Truncating to 3 decimals removes an amount uniform over one truncation bin: $(X_i - Y_i) \sim \text{Uni}(0, 0.001)$.

- (b) $X - Y = \sum_{i=1}^{1000} (X_i - Y_i)$ is a sum of 1000 i.i.d. RVs, so by the CLT it is approximately normal with

$$\mu = 1000 \cdot \frac{0.001}{2} = 0.5, \quad \sigma^2 = 1000 \cdot \frac{(0.001)^2}{12} = 8.33 \times 10^{-5}, \quad \sigma \approx 0.00913.$$

$$P(X - Y > 0.51) = 1 - \Phi\left(\frac{0.51 - 0.5}{0.00913}\right) = 1 - \Phi(1.10) \approx 0.137.$$

(The exam also accepted the two-tailed reading $P(|X - Y| > 0.51)$; the left tail $P(X - Y < -0.51)$ is essentially 0 here since the errors are all non-negative, so the answer is unchanged.)

Problem 6: Does the CLT Apply? (Fall 2023 Final)

No! The CLT requires the summed variables to be independent **and identically distributed**. These six are independent but come from six different distributions, so the CLT makes no claim about X . (With only $n = 6$ terms, “large n ” would also be a stretch.)

Problem 7: Rolling Until 300 (pset5: sum_dice)

(Problem set problem)

Challenge Problem: Sum of 30 Betas (pset5: sum_betas)

(Problem set problem)