CS109 Lecture 15 Worksheet: Adding Random Variables & the CLT

IID RVs, Convolution, Closed-Form Sums, and the Central Limit Theorem

Summer 2026

Lecture 15 Worksheet: Adding Random Variables & the CLT

Topics: IID random variables, the PMF of a sum (convolution), distributions with closed-form sums (Binomial, Poisson, Normal), sums of normals for zero-sum games (ELO), and the Central Limit Theorem with continuity correction.

How to use this sheet. Work in small groups. A TA and the instructor will circulate to help. We will go over selected solutions on the board. You do not need to finish every part; aim for understanding over speed. Keep fractions exact where you can.

Problem 1 (Review of Lecture 14): Click-Through Belief

A new "Subscribe" button is shown to 12 visitors and 9 click it. Your prior belief about the click probability X was $\text{Uni}(0, 1)$.

(a) What is your posterior belief about X?

(b) What is $E[X]$?

Problem 2: Sums with Closed Forms

For each part, name the distribution of the sum (with parameters), then answer the question. In all parts the variables are independent.

(a) $X \sim \text{Bin}(12,\ 0.25)$ and $Y \sim \text{Bin}(28,\ 0.25)$. What is the distribution of $X + Y$? Why does this argument require the two binomials to share the same p?

(b) Bugs arrive in your frontend code at rate $\lambda = 2.2/\text{week}$ and in your backend at rate $\lambda = 3.8/\text{week}$, independently. What is the distribution of the total bugs in a week, and what is $P(\text{total} = 5)$? (An expression is fine; it evaluates to about 0.161.)

(c) $X \sim N(10, 9)$ and $Y \sim N(20, 16)$. What is the distribution of $X + Y$, and what is $P(X + Y > 40)$?

Problem 3: Convolution by Hand

Let X and Y be independent, with PMFs:

$$P(X = 0) = 0.5, \quad P(X = 1) = 0.3, \quad P(X = 2) = 0.2; \qquad P(Y = 0) = 0.4, \quad P(Y = 1) = 0.6.$$

(a) Write the general convolution formula for $P(X + Y = n)$ for independent, non-negative, discrete X and Y.

(b) Use it to compute $P(X+Y=2)$.

(c) In one sentence: why do we sum over the ways to split n, and why does independence matter?

Problem 4: ELO and Zero-Sum Games

Two chess teams play one match. Team A has ELO score 1650 and team B has 1500. Following the ELO model from lecture, each team's performance ("ability") in the match is an independent normal: $A \sim N(1650,\ 200^2)$ and $B \sim N(1500,\ 200^2)$. Team A wins if $A > B$.

(a) Let $D = A - B$. What is the distribution of D? (Careful with the variance of a *difference*.)

(b) Compute $P(\text{A wins}) = P(D > 0)$.

(c) If both variances doubled (each team more erratic), would $P(\text{A wins})$ move toward 0.5 or away from it? Answer without computing.

Problem 5: Truncation Error (problem from the Winter 2025 Final)

Let X_i be an i.i.d. sample from Uniform$(0,1)$ with infinite precision. Let Y_i be the representation of X_i on a microcontroller: X_i truncated to 3 decimal places (e.g. $0.28374110\ldots \mapsto 0.283$).

(a) What is the distribution of $(X_i - Y_i)$, the error when a single uniformly sampled value is stored?

(b) We sample 1000 values; let $X = \sum_{i=1}^{1000} X_i$ and $Y = \sum_{i=1}^{1000} Y_i$. Using the CLT, approximate the probability that $X - Y$ is more than 0.51. (Recall Uni(α,β) has mean $\frac{\alpha+\beta}{2}$ and variance $\frac{(\beta-\alpha)^2}{12}$.)

Problem 6: Does the CLT Apply? (problem from the Fall 2023 Final)

Let $X = A+B+C+D+E+F$ where A through F are all independent random variables. Does the Central Limit Theorem apply to X? Explain as if teaching, in 1 or 2 sentences.

$$A \sim \text{Bernoulli}(p=0.5), \quad B \sim \text{Bin}(n=4, p=0.8), \quad C \sim \text{Geo}(p=0.3),$$

$$D \sim \text{Uni}(\alpha=0, \beta=1), \quad E \sim \text{Beta}(a=2, b=3), \quad F \sim \text{Exp}(\lambda=3).$$

Problem 7: Rolling Until 300 (pset5: sum_dice)

A fair 6-sided die is repeatedly rolled until the total sum of all the rolls exceeds 300. Approximate the probability that at least 80 rolls are necessary to reach a sum that exceeds 300. (Hint: turn the statement "at least 80 rolls are needed" into a statement about the sum of the first 79 rolls, then apply the CLT with a continuity correction. A single die roll has mean 7/2 and variance 35/12.)

Challenge Problem (optional): Sum of 30 Betas (pset5: sum_betas)

Let $X = \sum_{i=1}^{30} X_i$ where each $X_i \sim \text{Beta}(a=4, b=2)$, i.i.d. Use the CLT to approximate $P(19 < X < 20)$. (Recall $E[X_i] = \frac{a}{a+b}$ and $\text{Var}(X_i) = \frac{ab}{(a+b)^2(a+b+1)}$. No continuity correction here — why not?)