CS109A Week 3 Notes

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I. White Elephant

In a "white elephant" gift exchange[1], each person brings a gift, and then, one at a time, each person is randomly assigned one gift from the pile, possibly their own. We'll ignore the complicated rules in the real thing involving stealing.[2]

Problem 1. n people participate in a white elephant gift exchange.

(a) Suppose that you actually *hope* to get your own gift, like, you shelled out for some swanky Ferrero Rochers or something. The other players even agree to let you take your turn at any time, either right at the beginning or after anyone else's turn (and you get to see what gift each person gets up to the point that you jump in). Using the best strategy you can think of, what is the maximum possible probability that you get your own gift?

(b) What is the probability that all n people get their own gift?

(c) What is the probability that exactly $n-1$ people get their own gift?

(d) What is the probability that exactly $n-2$ people get their own gift?

(e) Suppose we want to know the probability that *nobody* gets their own gift (which is probably how these exchanges should actually work!) What is wrong with the following argument?

There is a $\frac{1}{n}$ chance that the first person gets their own gift. We are interested in the event where nobody gets their own gift, so the chances that we are still OK after the first person are $\frac{n-1}{n}$.

Now, in that case, that first person got someone else's gift. Consider that person B. They definitely did not get their own gift, so they got the gift of some other person C, and so on. So the overall chance is $\frac{n-1}{n}$.

(f) **Hard**: How *do* we find that probability? (Think recursively!)

[1] Apparently the name connotes a "gift" that is actually an expensive burden.

[2] A friend of mine does this with his family each December and says that it gets quite cutthroat.

Solutions to Problem 1.

(a) The other people weren't actually being nice... they knew that it doesn't matter where in the order we go, even if we get to see all the previous information! No matter when we choose to go, our chances of getting our own gift are exactly $\frac{1}{n}$.

The gifts will come out in *some* particular unknown order. The gift we brought is equally likely to appear first, second, ..., or n-th. It may seem like we can wait a little while for some gifts that aren't ours to go away, and then pounce, but this turns out not to help... or to hurt!

Consider the start of the game, for instance. We have to choose whether to go or wait:

- If we go first, we have a $\frac{1}{n}$ chance of getting our gift (winning). Otherwise we lose.
- If we wait and then go second, we only win if two things happen: the first person doesn't get our gift, and then our gift comes up for us. The probability of this is $\frac{n-1}{n} \cdot \frac{1}{n-1} = \frac{1}{n}$.

We can generalize this as follows. Our strategy consists only of choosing whether or not to go every time we have the chance to, and whenever we have a choice, it must be the case that our gift has not come up yet. So all strategies are really of the form "wait for k other people to go, then go". To win, we need our gift to not come up in the first k rounds, and the chances of this are $\frac{n-k}{n}$ (you can imagine the gifts being in some linear order chosen uniformly at random, and they come out in that order, so the chance of the gift being buried behind at least k others is $\frac{n-k}{n}$). Then, on our turn, there are $n-k$ gifts remaining, so our chance of getting our own gift is $\frac{1}{n-k}$. The overall win probability is therefore $\frac{n-k}{n} \cdot \frac{1}{n-k} = \frac{1}{n}$, regardless of the value of k.

(b) The probability that the first person gets their own gift is $\frac{1}{n}$, then (conditioned on that) the probability that the second person gets their own gift is $\frac{1}{n-1}$, and so on, so the answer is $\frac{1}{n \cdot n-1 \cdot \ldots \cdot 1} = \boxed{\frac{1}{n!}}$. Equivalently, there are $n!$ orderings in which the gifts can come out of the pile, and only one of those corresponds to everyone getting their own gift.

(c) If that one person doesn't get their own gift, then they must have someone else's gift, so that person didn't get theirs either. Therefore this situation is impossible: the probability is $\boxed{0}$.

(d) For exactly $n-2$ people to get their own gifts, exactly one pair must have their gifts swapped. There are $\binom{n}{2}$ ways to choose this pair, and once we have done that, everything else is forced (the two people in the pair get each other's gifts, and everyone else gets their own gift). So, given that there are $n!$ ways to distribute the gifts among the people, the answer is $\frac{\binom{n}{2}}{n!} = \boxed{\frac{1}{2(n-2)!}}$.

(e) The issue is as follows: we are saying that person A got the gift of some person B, who then must have in turn gotten the gift of some person C, and so on. But, for example, B could have gotten the gift of A.

However, this does suggest a way to solve the problem:

(f) Let $P(n)$ be the probability that nobody in a group of n gets their own gift. As base cases, $P(1) = 0$ (if there is only one person, of course they only get their own gift) and $P(2) = \frac{1}{2}$ (there is an equal chance of the people's gifts being swapped or not).

Otherwise, choose some arbitrary person A in the group. With probability $\frac{n-1}{n}$, they have the gift of some other person; call that person B. Now, one of two things is true:

- With probability $\frac{1}{n-1}$, B has A's gift, since A's gift is one of the $n-1$ remaining ones. In that case, we have an instance of the original problem for the remaining $n-2$ people.
- With probability $\frac{n}{n-1}$, B has the gift of some other person C...

and so on. So we can write

$P(n) = \frac{n-1}{n}(\frac{1}{n-1}P(n-2) + \frac{n-2}{n-1}(\frac{1}{n-2}P(n-3) + \frac{n-3}{n-2}...$

$= (\frac{n-1}{n})(\frac{1}{n-1})P(n-2) + (\frac{n-1}{n})(\frac{n-2}{n-1})(\frac{1}{n-2})P(n-3) + ...$

$= \frac{1}{n}P(n-2) + \frac{1}{n}P(n-3) + ... + \frac{1}{n}P(2) + \frac{1}{n}P(1) + \frac{1}{n}P(0).$

$= \frac{1}{n}\sum_{k=0}^{n-2} P(k).$

Exploring this pattern, we see that

- $P(3) = \frac{1}{3}(1+0) = \frac{1}{3}$
- $P(4) = \frac{1}{4}(1+0+\frac{1}{2}) = \frac{3}{8}$
- $P(5) = \frac{1}{5}(1+0+\frac{1}{2}+\frac{1}{3}) = \frac{11}{30}$

and on and on. This does not boil down to any nice form that we have seen so far, but at least we now have a recursive way of calculating the answer for any n.[3]

[3]If you want to explore more: a permutation in which no value is at its own position in the list – e.g., 2 3 1 5 4 – is called a *derangement*. A value that is at its own position – e.g., the 3 in 5 1 3 2 4 – is called a *fixed point*.

II. Independence

Suppose we had asked a different question about the white elephant scenario in Problem 1: what is the *expected* number of people who get their own gift?

Since an expectation is essentially a weighted average, our first instinct might be to write out an expression like the following (where X is a random variable for the number of people who get their own gift):

$\mathbf{E}(X) = (0)(P(X = 0)) + (1)(P(X = 1)) + ... + (n)(P(X = n))$

This is correct, but we have a different problem: it is annoying to compute. We have seen how hard it is even to find just the value of $P(X = 0)$.[4] It just isn't going to be tractable to try to do the same for each of $P(X = 1)$, etc.

One of the main difficulties we struggled with in the white elephant problem was a lack of independence. Informally, whether person A gets their own gift is at least somewhat tied to whether some other person B gets their own gift! That is, if we are A and we know that B did not get their own gift, then they might have gotten ours instead... but if B did get their own gift, ours is definitely still out there!

Let E be the event that Person A gets their gift and F be the event that Person B gets their gift. One general definition of independence of two events is $P(E|F) = P(E)$. We can see, for general n, that $P(E) = \frac{1}{n}$, but $P(E|F) = \frac{1}{n-1}$, since now Person A does not need to worry about getting Person B's gift.

Another (equivalent) definition of independence of two events is $P(E \cap F) = P(E)P(F)$. Consider the $n = 2$ case, for example. Since there are only two possible outcomes (either A and B each get their own gifts, or each one gets the other's gift), we see that $P(E) = \frac{1}{2}, P(F) = \frac{1}{2}, P(E \cap F) = \frac{1}{2}$, and $\frac{1}{2} \neq \frac{1}{2} \cdot \frac{1}{2}$.

Problem 2. Let's get a bit more practice with independence. We flip one penny and one nickel. Consider the following events:

- A: The penny comes up tails.
- B: The nickel comes up tails.
- C: Neither coin comes up tails.
- D: Exactly one of the coins comes up tails.

Which of these pairs of events, if any, are independent?

[4] I wanted to put an exclamation point here, but when writing about probability, I have to make sure my emphatic !s aren't confused for factorials!

Solution to Problem 2. We'll look at all of the $\binom{4}{2} = 6$ pairs individually.

- A and B: $P(A \cap B) = \frac{1}{4}, P(A) = \frac{1}{2}, P(B) = \frac{1}{2}$, so these are independent, as they intuitively must be. (We never said that the penny flip influences – or is influenced by – the nickel flip, in any way.)

- A and C: These events are mutually exclusive, and mutually exclusive events cannot be independent. Just to back it up with math, though, $P(A \cap C) = 0$, and $P(A)P(C) = \frac{1}{2} \cdot \frac{1}{4} \neq 0$. Alternatively: knowing that one of A or C has happened makes us certain that the other has not, so they cannot possibly be independent, i.e. $P(A|C) = 0 \neq P(A)$.

- A and D: Perhaps surprisingly, these are independent, even though D is clearly related to / influenced by A. But what matters is whether this influence is disproportionate, and here it is not: $P(D|A) = \frac{P(D \cap A)}{P(A)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2} = P(D)$.

  More intuititively, suppose that before we see the results of any flips, we first calculate the probability of D as $\frac{2}{4}$ (either the penny comes up tails and the nickel comes up heads, or vice versa). Then we are told that the penny has come up tails. Does this information change our belief about D? The nickel has a $\frac{1}{2}$ chance of being tails, so D still has probability $\frac{1}{2}$.

  It *is* true in general that if two events are clearly not related to each other, they are independent. But just because two events are related to each other does not necessarily mean that they are not independent!

- Others: For B and C, B and D, there is nothing special about the nickel vs. the penny, so these results must be the same as for A and C and A and D. For C and D, again, these events are mutually exclusive.

Therefore the only independent pairs are $\boxed{(A, B), (A, D), (B, D)}$.

III. Linearity of Expectation

Let's keep working on the expected number of people in the white elephant gift exchange who get their own gift. We'll use some powerful tools that doesn't get a ton of screen time in CS109, but are very useful to know in general.[5]

First, *linearity of expectation.* Let X and Y be random variables. Then $\mathbf{E}(X + Y) = \mathbf{E}(X) + \mathbf{E}(Y)$.

But wait – we never said whether X and Y were independent. We just spent a lot of time fussing over that. Surely it matters? *Nope.* The statement above follows directly from the definition of expectation. Let's illustrate this with our

[5]It comes up in CS161, for instance, where they assume you know it from 109.

familiar example of rolling two six-sided dice and taking the sum. Let X and Y be the results of the first and second rolls, and let $X+Y$ be their combined sum.

We can find $\mathbf{E}(X)$ (or $\mathbf{E}(Y)$) directly: it is a weighted average, a sum in which each term is one of the possible values times the probability of that value. In this case, it is $\frac{1}{6}(1)+\frac{1}{6}(2)+\frac{1}{6}(3)+\frac{1}{6}(4)+\frac{1}{6}(5)+\frac{1}{6}(6) = \frac{7}{2}$. Notice that because all of the outcomes have equal probabilities, this is just the same as taking an arithmetic mean, but this is *not* true of expectations in general, as we're about to see!

What about $\mathbf{E}(X+Y)$? Again, let's directly take an expectation as a weighted average. Recall our probability distribution from Week 1:

2	3	4	5	6	7	8	9	10	11	12
$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

To find the expectation, we multiply each value by its probability and then add those terms together, i.e., $\frac{1}{36}\cdot 2+\frac{2}{36}\cdot 3+\ldots+\frac{2}{36}\cdot 11+\frac{1}{36}\cdot 12$. This ends up being 7. (We could have also noticed that the distribution is symmetric around 7.)

So we have shown that in this case, at least, $\mathbf{E}(X+Y) = \mathbf{E}(X)+\mathbf{E}(Y)$. What's the intuition? Let's use a slightly different version of a table from Week 1:

	1	**2**	**3**	**4**	**5**	**6**
1	1 + 1	1 + 2	1 + 3	1 + 4	1 + 5	1 + 6
2	2 + 1	2 + 2	2 + 3	2 + 4	2 + 5	2 + 6
3	3 + 1	3 + 2	3 + 3	3 + 4	3 + 5	3 + 6
4	4 + 1	4 + 2	4 + 3	4 + 4	4 + 5	4 + 6
5	5 + 1	5 + 2	5 + 3	5 + 4	5 + 5	5 + 6
6	6 + 1	6 + 2	6 + 3	6 + 4	6 + 5	6 + 6

When we take $\mathbf{E}(X+Y)$, we are adding up all of the orange and the blue numbers in the 36 cells of the table and then dividing by 36. But this is the same as separately adding up all of the orange numbers divided by 36 and all of the blue numbers divided by 36, which is the same as taking $\mathbf{E}(X)+\mathbf{E}(Y)$. So these two expressions are just adding up the same numbers in different ways.[6] And we never used or needed independence; we are just adding up cells in a table!

Problem 3. In parts (a) and (b), let X and Y be die rolls as above.

(a) What is $\mathbf{E}(X^2)$? (Use the definition of expectation.)

(b) What is $\mathbf{E}(2X-Y)$? (Use linearity of expectation!)

(c) Hm, is there a similar magical thing for products? Can you find two random variables A, B for which $\mathbf{E}(AB) = \mathbf{E}(A)\mathbf{E}(B)$?

(d) But wait... can you find two random variables C, D for which $\mathbf{E}(CD) \neq \mathbf{E}(C)\mathbf{E}(D)$?

[6] Even a mathematically rigorous proof of linearity of expectation boils down to just this.

Solutions to Problem 3.

(a) X^2 is just a random variable with the following distribution. (Think of replacing each number on the die with the square of that number.)

1^2	2^2	3^2	4^2	5^2	6^2
$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$

So the expectation is $\frac{1}{6} \cdot 1^2 + \frac{1}{6} \cdot 2^2 + ... + \frac{1}{6} \cdot 6^2 = \frac{1+4+9+16+25+36}{6} = \boxed{\frac{91}{6}}$.

A common error I saw last quarter with this sort of thing was to square the probabilities as well as the values. But then notice what happens: the sum of the probabilities becomes $\frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} = \frac{1}{6}$, which is not 1. So we would no longer even have a valid probability distribution!

(b) We could write out a table of all these values, but it is so much faster and nicer to use linearity of expectation:

- $2X$ is just a new random variable with the distribution

2	4	6	8	10	12
$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

  So its expectation is just twice the expectation of X, i.e. 7. In general, $\mathbf{E}(cX)$, for some scalar multiple c, is $c\mathbf{E}(X)$.
- Similarly, $-Y$ is a random variable with the same distribution as Y, but with all the values negative. So its expectation is $-1 \cdot \mathbf{E}(Y) = -\frac{7}{2}$.
- We can use linearity of expectation with $2X$ and $-Y$ as with any random variables: the expectation of their sum is the sum of their expectations. $\mathbf{E}(2X - Y) = \mathbf{E}(2X) + \mathbf{E}(-Y) = 7 - \frac{7}{2} = \boxed{\frac{7}{2}}$.

(c) This actually works with X and Y above, for example! To make things simpler, let's make them both 2-sided dice. Then $\mathbf{E}(X) = \mathbf{E}(Y) = \frac{3}{2}$, and $\mathbf{E}(XY) = \frac{1}{4} \cdot 1 + \frac{1}{2} \cdot 2 + \frac{1}{4} \cdot 4 = \frac{9}{4}$. And $\frac{9}{4} = \frac{3}{2} \cdot \frac{3}{2}$.

(d) Before we get too excited, though... your intuition may tell you that we might run into issues with independence here. After all, we can only get away with multiplying probabilities when the corresponding events are independent! Let's keep thinking about two-sided dice, and let X be the result of a roll, and Y be 3 minus *that same roll.* Then we have

- $\mathbf{E}(X) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 2 = \frac{3}{2}$
- $\mathbf{E}(Y) = \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 1 = \frac{3}{2}$
- $\mathbf{E}(XY) = \frac{1}{2} \cdot (1)(2) + \frac{1}{2} \cdot (2)(1) = \frac{1}{2}$. But $\frac{1}{2} \neq \frac{3}{2} \cdot \frac{3}{2}$.

It turns out that $\mathbf{E}(XY) = \mathbf{E}(X)\mathbf{E}(Y)$ can still be useful: it is true precisely when X and Y are independent. We will not prove that here, but it can be shown using the definition of expectation.

IV. Indicator Random Variables

We haven't yet seen why linearity of expectation is awesome, or how it can help us solve our white elephant problem. To do the latter, we need one final tool: indicator random variables.

Every event A has a corresponding *indicator random variable* $\mathbf{I}_A$, defined to be 1 when A is true, and 0 when A is false. For instance, if A is the event that a fair coin flip comes up heads, then $\mathbf{I}_A$ is 1 half the time and 0 half the time. In general we see that

$P(\mathbf{I}_A = 1) = P(A),$

i.e. that the probability of an indicator variable being 1 is just the probability of the underlying event happening.

What about the expectation of an indicator variable? $\mathbf{E}(I_A) = P(A) \cdot 1 + P(A^c) \cdot 0 = P(A)$.

So what? Well, let's define some indicator variables for our gift exchange. Let A_1 be the event that person 1 gets their own gift. Let A_2, ..., A_n be defined likewise for persons 2 through n. Then consider the corresponding indicator variables $I_1, ..., I_n$.

The number of people who get their own gifts is just the sum of $I_1 + ... + I_n$. This is because I_j is 1 when person j gets their gift back, and 0 otherwise, so we are just counting up the number of people who got their gifts.

So, to get the expected number of people who get their own gifts, we can take the expectation of this sum:

$\mathbf{E}(I_1 + ... + I_n)$

But then by linearity of expectation this is

$\mathbf{E}(I_1) + ... + \mathbf{E}(I_n)$

which, from our expression above for the expectation of an indicator, is

$P(A_1) + ... + P(A_n)$

Since $P(A_j) = \frac{1}{n}$ for each j, this becomes $n \cdot \frac{1}{n} = 1$. So in expectation, one person will get their own gift. Strangely, this does not even depend on the size of n.

Also noticed that we completely sidestepped all of the issues with non-independence from earlier. Unlike in our dice example, here I_1, ..., I_n were definitely not independent, because their underlying events $A_1, ..., A_n$ were definitely not independent. But linearity of expectation doesn't care!

There are some limitations on the usefulness of what we found, though. Let's think about what we can infer from an expectation in general.

Problem 4. Suppose that all we know about some random variable R is that it takes on nonnegative integer values, and has expectation 10. Which of the following *must* be true?

(a) $P(R = 10) \geq P(R = j)$, for any individual $j \neq 10$. (That is, 10 is the/a most common value.)

(b) $P(R \leq 10) \geq 0.5$. (That is, at least half of the mass of the distribution of R lies at or below 10.)

(c) $P(R = 10) > 0$.

(d) $P(R \geq 20) \leq \frac{1}{2}$.

Also, here's some further practice with this idea:

Problem 5. Suppose that we are studying strands of messenger RNA (mRNA). For the purposes of this problem, just think of mRNA as a string of letters in which every letter is either `A`, `C`, `G`, or `U`.

We are interested in instances of the pattern `AGUA`. Notice that these may overlap, so the string `GUAGUAGUACAGUAGU` has two instances of the pattern, starting at the 3rd, 6th, and 11th positions, respectively.

Suppose that each letter of mRNA is generated completely randomly and independently, with probability $\frac{1}{3}$ for `A`, $\frac{1}{6}$ for `C`, $\frac{1}{6}$ for `G`, and $\frac{1}{3}$ for `U`.[7]

Consider a string of length 8. Let V_i be the event of seeing `AGUA` starting at position 1.

(a) What is $P(V_1)$?

(b) What is $P(V_2)$?

(c) Are V_1 and V_2 independent?

(d) Are V_1 and V_4 independent?

(e) Are V_1 and V_5 independent?

(f) What is the expected number of instances of `AGUA` in this string?

[7]This is biologically rather unlikely, I know. At least I made P(`A`) = P(`U`) and P(`C`) = P(`G`).

Solutions to Problem 4.

(a) False. The mean (average/expectation) is not necessarily the mode (most common value). For instance, R could be 8 half the time and 12 the other half of the time, and it would still have expectation 10.

(b) False. The mean is not necessarily the median. For instance, R could be 11 with probability 0.9 and 1 with probability 0.1 (you may want to check that the expectation is still 10). In this case, $P(R \leq 10) = 0.1$.

(c) False, as we saw from our counterexample to part (a).

(d) True! Suppose this were false, i.e., $P(R \geq 20) > \frac{1}{2}$. Then R would have a value of at least 20 strictly more than half the time. But then R's expectation would be strictly greater than $\frac{1}{2} \cdot 20 = 10$, and we already know the expectation is exactly 10.

A more general expression of this idea is *Markov's inequality*: for a non-negative random variable R, for any positive constant c, $P(R \geq c) \leq \frac{\mathbf{E}(R)}{c}$. This is a weak bound, but it is surprisingly useful in theoretical CS.

Solutions to Problem 5.

(a) $P(V_1) = P(\texttt{A} \text{ in position } 1) \cdot P(\texttt{G} \text{ in position } 2) \cdot P(\texttt{U} \text{ in position } 3) \cdot P(\texttt{A} \text{ in position } 4)$, since these are independent. So the answer is $\frac{1}{3} \cdot \frac{1}{6} \cdot \frac{1}{3} \cdot \frac{1}{3} = \boxed{\frac{1}{162}}$.

(b) The answer here is also $\boxed{\frac{1}{162}}$, by exactly the same reasoning. The argument in (a) didn't depend on where we started; because the individual letters get generated independently, their surroundings don't matter.

(c) No. The presence of `AGUA` at positions 1 through 4 would preclude another `AGUA` from starting at position 2, so $P(V_2|V_1) = 0 \neq P(V_2)$.

(d) No. The presence of `AGUA` at positions 1 through 4 actually makes it *more* likely for another `AGUA` to start at position 4, since that latter `AGUA` gets a head start with its `A` already in place. That is, $P(V_4|V_1) > P(V_4)$.

(e) Yes. Since the `AGUA` at positions 1-4 makes an `AGUA` at positions 4-7 more likely, it seems like this should make an `AGUA` at positions 5-8 less likely. But this kind of argument is dangerously narrow – we could just as easily say that the `AGUA` at positions 1-4 makes it impossible for `AGUA`s to start at position 3, which decreases the chance of a `U` appearing in position 5, which should increase the probability of an `AGUA` there!

Put another way, if someone flipped 4 coins and told you the first three were TT, you might reasonably say "I now believe the second through third are more likely to be TT", but it wouldn't make sense to say "I now believe the third through fourth are more likely to be TT". Indeed, even knowing that the first two are TT, there are still four equally likely possibilities: TTHH, TTHT, TTTH, TTTT. So the last two are equally likely to be HH, HT, TH, TT.

(f) Here we use linearity of expectation! Consider the indicator variables $I_1, I_2, ..., I_5$ corresponding to the events $V_1, V_2, ..., V_5$. (We don't consider V_6 through V_8 because there is not enough room for an `AGUA` to start there.)

The total number of instances of AGUA as $\mathbf{E}(I_1 + I_2 + I_3 + I_4 + I_5)$. Even though e.g. I_1 and I_2 are not independent, we can still use linearity of expectation to turn this into $\mathbf{E}(I_1) + \mathbf{E}(I_2) + \mathbf{E}(I_3) + \mathbf{E}(I_4) + \mathbf{E}(I_5)$, which equals $P(V_1) + P(V_2) + P(V_3) + P(V_4) + P(V_5)$. Since each of those is $\frac{1}{162}$, the overall answer is $\boxed{\frac{5}{162}}$.

V. Card Hands

On Homework 1, some CS109 students presumably got the three of a kind problem correct using at least some trial and error – dividing by an extra factor of 2, etc. But it's good to build up confidence for the exam, where you only get to give one answer. Suppose we want to find the probability of drawing a full house (three cards of one rank and two cards of another rank).[8]

When we solve a card problem, the first decision we need to make is whether each outcome in the sample space is an unordered set of cards, or an ordered list of cards. The choice is up to us and not an inherent feature of the problem.

One option: Outcomes are unordered sets

In this case, the sample space is always $\binom{52}{5}$: the number of ways to choose an unordered set of 5 out of 52 things.

What about the event space? To build a valid hand, we want three cards of one rank and two cards of another rank. We have $\binom{13}{1}$ choices for the rank of the three-card set, and then we want any three of the four cards of that suit: $\binom{4}{3}$. Then (conditioned on our choice of the first rank) we have $\binom{12}{1}$ choices for the rank of the two-card set, and we want any two of the four cards of that suit: $\binom{4}{2}$. So the number of sets of cards that are full houses is $\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}$.[9]

[8] In memory of the great Bob Saget...

[9] What if we had chosen the two ranks at the outset, i.e. $\binom{13}{2}$, and then chosen one of those to be the rank of the three-card set, i.e. $\binom{2}{1}$? That also works! $\binom{13}{2}\binom{2}{1}$ is the same as $\binom{13}{1}\binom{12}{1}$; both work out to be $13 \cdot 12$.

Therefore the overall probability is $\frac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}{\binom{52}{5}}$, which works out to $\frac{6}{4165}$.

Another option: Outcomes are ordered lists

- There are 52 ways to draw our first card.
- Let's choose a card that matches the rank of our first card. There are only 3 such cards left in the deck, out of 51.
- Now let's complete the set of three cards of the same rank. There are only 2 valid cards remaining, out of 50.
- Now, to build our set of 2, we can start with any remaining card in the deck that doesn't have that previous rank. There are 48 such cards left, out of 49.
- Now we need one of the 3 remaining cards (out of 48) with the same rank as our previous choice.

So here we would get an answer of $\frac{52 \cdot 3 \cdot 2 \cdot 48 \cdot 3}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48} = \frac{3}{20825}$. Oh no, this isn't the same as the previous answer... it is 10 times smaller! What went wrong?

The problem is that here we said the order of the cards in the hand matters, but then we required the set of 3 to come before the set of 2. I.e., letting A represent the rank repeated thrice and B represent the rank repeated twice, we enforced the pattern AAABB and got $52 \cdot 3 \cdot 2 \cdot 48 \cdot 3$. But we could have just as easily chosen ABABA, for instance, and gotten $52 \cdot 48 \cdot 3 \cdot 3 \cdot 2$. Specifically, there are $\binom{5}{2} = 10$ such patterns, which accounts for our missing factor of 10.

I personally find the ordered list method more error-prone, but you should use whatever method you "click" most with.

Problem 6. One dirty secret of Vegas is that the resorts are pretty much all the same. Suppose that one of them tries to spice things up by introducing six-card poker! The game uses a standard deck, but with hands of six cards. (All of these descriptions are of unordered sets of cards.)

(a) A "three pair" is a hand of two cards of one rank, two cards of another rank, and two cards of a third rank. What is the probability of drawing a three pair?

(b) A "double straight" is two cards of one rank, followed by two cards of the next highest rank, followed by two cards of the next highest rank after that. What is the probability of drawing a double straight? (In CS109, straights can use an ace as a low card.)

(c) A "pyramid" is three cards of one rank, two cards of another rank, and one card of a third rank. What is the probability of drawing a pyramid?

Solutions to Problem 6. Let's do this treating outcomes as unordered sets rather than ordered lists.

(a)
- Choose the three ranks involved: $\binom{13}{3}$.
- For each rank, choose the suits of the two cards: $\binom{4}{2}^3$.

Therefore the answer is $\frac{\binom{13}{3}\binom{4}{2}^3}{\binom{52}{6}} = \boxed{\frac{594}{195755}}$.

(b) First, there are 12 possible ranks that a straight of 3 cards can start on: A, 2, ..., Q. Once we pick the starting rank, it forces the ranks of all of the cards. E.g., picking 9 as the starting rank forces the other two ranks to be 10 and J.

Now, for each of the three ranks involved, our two cards of that rank can have any two of the four suits. So the answer is $\frac{12\cdot\binom{4}{2}^3}{\binom{52}{6}} = \boxed{\frac{324}{2544815}}$.[10]

(c) Now we must pick one rank for the set of three, one of the remaining ranks for the set of two, one of the remaining ranks for the remaining lone card. Then, when choosing suits, we pick three of the four cards of the first rank, two of the four cards of the second rank, and one of the four cards of the third rank. This is $\frac{\binom{13}{1}\binom{12}{1}\binom{11}{1}\binom{4}{3}\binom{4}{2}\binom{4}{1}}{\binom{52}{6}} = \boxed{\frac{1584}{195755}}$.

[10] Note that this hand is actually a subset of the previous one. If this were an actual casino game, the rules would have to specify that "three pair" does not include "double straight"s.