CS156: The Calculus of Computation

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Chapter 4: Induction

Induction

- Stepwise induction (for T_{PA}, T_{cons})

- Complete induction (for T_{PA}, T_{cons})

  Theoretically equivalent in power to stepwise induction,
  but sometimes produces more concise proof

- Well-founded induction

  Generalized complete induction

- Structural induction

  Over logical formulae

Stepwise Induction (Peano Arithmetic T_{PA})

Axiom schema (induction)

$F[0] \wedge$	... base case
$(\forall n.\ F[n] \rightarrow F[n+1])$	... inductive step
$\rightarrow \forall x.\ F[x]$	... conclusion

for Σ_{PA}-formulae $F[x]$ with one free variable x.

To prove $\forall x.\ F[x]$, the conclusion, i.e.,

$F[x]$ is T_{PA}-valid for all $x \in \mathbb{N}$,

it suffices to show

- base case: prove $F[0]$ is T_{PA}-valid.
- inductive step: For arbitrary $n \in \mathbb{N}$,
  assume inductive hypothesis, i.e.,
  $F[n]$ is T_{PA}-valid,
  then prove
  $F[n+1]$ is T_{PA}-valid.

Example

Prove:

$$F[n] : 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

for all $n \in \mathbb{N}$.

- *Base case*: $F[0] : 0 = \frac{0 \cdot 1}{2}$
- *Inductive step*: Assume $F[n] : 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$, (IH) show

$$\begin{aligned} F[n+1] \quad &: \quad 1 + 2 + \cdots + n + (n+1) \\ &= \frac{n(n+1)}{2} + (n+1) && \text{by } (IH) \\ &= \frac{n(n+1) + 2(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2} \end{aligned}$$

Therefore,

$$\forall n \in \mathbb{N}.\ 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$

Example:

Theory T_{PA}^{+} obtained from T_{PA} by adding the axioms:

- $\forall x.\ x^0 = 1$ (E0)
- $\forall x, y.\ x^{y+1} = x^y \cdot x$ (E1)
- $\forall x, z.\ exp_3(x, 0, z) = z$ (P0)
- $\forall x, y, z.\ exp_3(x, y+1, z) = exp_3(x, y, x \cdot z)$ (P1)

($exp_3(x, y, z)$ stands for $x^y.z$)

Prove that

$$\boxed{\forall x, y.\ exp_3(x, y, 1) = x^y}$$

is T_{PA}^{+}-valid.

First attempt:

$$\forall y\ [\underbrace{\forall x.\ exp_3(x, y, 1) = x^y}_{F[y]}]$$

We chose induction on y. Why?

Base case:

$F[0]:\ \forall x.\ exp_3(x, 0, 1) = x^0$

For arbitrary $x \in \mathbb{N}$, $exp_3(x, 0, 1) = 1$ (P0) and $x^0 = 1$ (E0).

Inductive step: Failure.

For arbitrary $n \in \mathbb{N}$, we cannot deduce

$F[n+1] : \forall x.\ exp_3(x, n+1, 1) = x^{n+1}$

from the inductive hypothesis

$F[n] : \forall x.\ exp_3(x, n, 1) = x^n$

Second attempt: Strengthening

Strengthened property

$$\boxed{\forall x, y, z.\ exp_3(x, y, z) = x^y \cdot z}$$

Implies the desired property (choose $z = 1$)

$$\forall x, y.\ exp_3(x, y, 1) = x^y$$

Proof of strengthened property:

Again, induction on y

$$\forall y\ [\underbrace{\forall x, z.\ exp_3(x, y, z) = x^y \cdot z}_{F[y]}]$$

Base case:

$$F[0] :\ \forall x, z.\ exp_3(x, 0, z) = x^0 \cdot z$$

For arbitrary $x, z \in \mathbb{N}$, $exp_3(x, 0, z) = z$ (P0) and $x^0 = 1$ (E0).

Inductive step: For arbitrary $n \in \mathbb{N}$

Assume inductive hypothesis

$$F[n] : \forall x, z.\ exp_3(x, n, z) = x^n \cdot z \qquad \text{(IH)}$$

prove

$$F[n+1] : \forall x', z'.\ exp_3(x', n+1, z') = x'^{n+1} \cdot z'$$

$\uparrow$note

Consider arbitrary $x', z' \in \mathbb{N}$:

$$\begin{aligned} exp_3(x', n+1, z') &= exp_3(x', n, x' \cdot z') && \text{(P1)} \\ &= x'^n \cdot (x' \cdot z') && \text{IH } F[n]; x \mapsto x', z \mapsto x' \cdot z' \\ &= x'^{n+1} \cdot z' && \text{(E1)} \end{aligned}$$

Stepwise Induction (Lists T_{cons})

Axiom schema (induction)

$$
\begin{array}{ll}
(\forall \text{ atom } u.\ F[u]) \land & \ldots \text{ base case} \\
(\forall u, v.\ F[v] \ \rightarrow \ F[\text{cons}(u, v)]) & \ldots \text{ inductive step} \\
\ \rightarrow \ \forall x.\ F[x] & \ldots \text{ conclusion}
\end{array}
$$

for Σ_{cons}-formulae $F[x]$ with one free variable x.

Note: $\forall$ atom $u.\ F[u]$ stands for $\forall u.\ (\text{atom}(u) \ \rightarrow \ F[u])$.

To prove $\forall x.\ F[x]$, i.e.,

$F[x]$ is T_{cons}-valid for all lists x,

it suffices to show

- base case: prove $F[u]$ is T_{cons}-valid for arbitrary atom u.
- inductive step: For arbitrary lists u, v,
  assume inductive hypothesis, i.e.,
  $F[v]$ is T_{cons}-valid,
  then prove
  $F[\text{cons}(u, v)]$ is T_{cons}-valid.

Example: Theory T^+_{cons} I

T_{cons} with axioms

Concatenating two lists

- $\forall$ atom $u.\ \forall v. concat(u, v) = cons(u, v)$ (C0)
- $\forall u, v, x.\ concat(\text{cons}(u, v), x) = \text{cons}(u, concat(v, x))$ (C1)

Example: Theory T^{+}_{cons} II

Example: for atoms a, b, c, d,

$$
\begin{aligned}
& concat(\text{cons}(a, \text{cons}(b, c)), d) & \\
= \ & \text{cons}(a, concat(\text{cons}(b, c), d)) & (C1) \\
= \ & \text{cons}(a, \text{cons}(b, concat(c, d))) & (C1) \\
= \ & \text{cons}(a, \text{cons}(b, \text{cons}(c, d))) & (C0)
\end{aligned}
$$

$$
\begin{aligned}
& concat(\text{cons}(\text{cons}(a, b), c), d) & \\
= \ & \text{cons}(\text{cons}(a, b), concat(c, d)) & (C1) \\
= \ & \text{cons}(\text{cons}(a, b), \text{cons}(c, d)) & (C0)
\end{aligned}
$$

Example: Theory T^{+}_{cons} III

Reversing a list

- $\forall$ atom $u.\ rvs(u) = u$ (R0)
- $\forall x, y.\ rvs(concat(x, y)) = concat(rvs(y), rvs(x))$ (R1)

Example: for atoms a, b, c,

$$
\begin{array}{lll}
 & rvs(\text{cons}(a, \text{cons}(b, c))) & \\
= & rvs(concat(a, concat(b, c))) & (C0) \\
= & concat(rvs(concat(b, c)), rvs(a)) & (R1) \\
= & concat(concat(rvs(c), rvs(b)), rvs(a)) & (R1) \\
= & concat(concat(c, b), a) & (R0) \\
= & concat(\text{cons}(c, b), a) & (C0) \\
= & \text{cons}(c, concat(b, a)) & (C1) \\
= & \text{cons}(c, \text{cons}(b, a)) & (C0)
\end{array}
$$

Example: Theory T^+_{cons} IV

Deciding if a list is flat;

i.e., $\mathit{flat}(x)$ is true iff every element of list x is an atom.

- $\forall$ atom $u.\ \mathit{flat}(u)$ (F0)
- $\forall u, v.\ \mathit{flat}(\text{cons}(u, v)) \leftrightarrow \text{atom}(u) \wedge \mathit{flat}(v)$ (F1)

Example: for atoms a, b, c,

$$\begin{aligned} \mathit{flat}(\text{cons}(a, \text{cons}(b, c))) &= \text{true} \\ \mathit{flat}(\text{cons}(\text{cons}(a, b), c)) &= \text{false} \end{aligned}$$

Prove

$$\boxed{\forall x.\ \underbrace{flat(x) \ \rightarrow \ rvs(rvs(x)) = x}_{F[x]}}$$

is T_{cons}^{+}-valid.

Base case: For arbitrary atom u,

$$F[u] : flat(u) \ \rightarrow \ rvs(rvs(u)) = u$$

by $F0$ and R0.

Inductive step: For arbitrary lists u, v, assume the inductive hypothesis

$$F[v] : \ flat(v) \ \rightarrow \ rvs(rvs(v)) = v \qquad \text{(IH)}$$

and prove

$$F[\text{cons}(u,v)] \quad : \quad \mathit{flat}(\text{cons}(u,v)) \rightarrow$$
$$\mathit{rvs}(\mathit{rvs}(\text{cons}(u,v))) = \text{cons}(u,v) \qquad (*)$$

<u>Case $\neg\text{atom}(u)$</u>

$$\mathit{flat}(\text{cons}(u,v)) \Leftrightarrow \text{atom}(u) \wedge \mathit{flat}(v) \Leftrightarrow \perp$$

by (F1). $(*)$ holds since its antecedent is $\perp$.

<u>Case $\text{atom}(u)$</u>

$$\mathit{flat}(\text{cons}(u,v)) \Leftrightarrow \text{atom}(u) \wedge \mathit{flat}(v) \Leftrightarrow \mathit{flat}(v)$$

by (F1). Now, show

$$\mathit{rvs}(\mathit{rvs}(\text{cons}(u,v))) = \cdots = \text{cons}(u,v).$$

Missing steps:

$$
\begin{aligned}
& rvs(rvs(\text{cons}(u,v))) & \\
= \;& rvs(rvs(concat(u,v))) & \text{(C0)} \\
= \;& rvs(concat(rvs(v), rvs(u))) & \text{(R1)} \\
= \;& concat(rvs(rvs(u)), rvs(rvs(v))) & \text{(R1)} \\
= \;& concat(u, rvs(rvs(v))) & \text{(R0)} \\
= \;& concat(u, v) & \text{(IH), since } flat(v) \\
= \;& \text{cons}(u, v) & \text{(C0)}
\end{aligned}
$$

Complete Induction (Peano Arithmetic T_{PA})

Axiom schema (complete induction)

$$(\forall n.\ \underbrace{(\forall n'.\ n' < n \ \rightarrow\ F[n'])}_{IH} \ \rightarrow\ F[n]) \qquad \ldots \text{ inductive step}$$

$$\rightarrow\ \forall x.\ F[x] \qquad \ldots \text{ conclusion}$$

for Σ_{PA}-formulae $F[x]$ with one free variable x.

To prove $\forall x.\ F[x]$, the conclusion i.e.,

$F[x]$ is T_{PA}-valid for all $x \in \mathbb{N}$,

it suffices to show

- inductive step: For arbitrary $n \in \mathbb{N}$,
  assume inductive hypothesis, i.e.,
  $F[n']$ is T_{PA}-valid for every $n' \in \mathbb{N}$ such that $n' < n$,
  then prove
  $F[n]$ is T_{PA}-valid.

Is base case missing?

No. Base case is implicit in the structure of complete induction.

Note:

- Complete induction is theoretically equivalent in power to stepwise induction.
- Complete induction sometimes yields more concise proofs.

Example: Integer division $\quad quot(5,3) = 1$ and $rem(5,3) = 2$

Theory T_{PA}^{*} obtained from T_{PA} by adding the axioms:

- $\forall x, y.\ x < y \ \rightarrow\ quot(x, y) = 0 \quad$ (Q0)
- $\forall x, y.\ y > 0 \ \rightarrow\ quot(x + y, y) = quot(x, y) + 1 \quad$ (Q1)
- $\forall x, y.\ x < y \ \rightarrow\ rem(x, y) = x \quad$ (R0)
- $\forall x, y.\ y > 0 \ \rightarrow\ rem(x + y, y) = rem(x, y) \quad$ (R1)

Prove

(1) $\forall x, y.\ y > 0 \ \rightarrow\ rem(x, y) < y$

(2) $\forall x, y.\ y > 0 \ \rightarrow\ x = y \cdot quot(x, y) + rem(x, y)$

Best proved by complete induction.

Proof of (1)

$$\forall x.\ \underbrace{\forall y.\ y > 0 \ \rightarrow\ rem(x, y) < y}_{F[x]}$$

Consider an arbitrary natural number x.

Assume the inductive hypothesis

$$\forall x'.\ x' < x \ \rightarrow\ \underbrace{\forall y'.\ y' > 0 \ \rightarrow\ rem(x', y') < y'}_{F[x']} \qquad \text{(IH)}$$

Prove $\quad F[x] : \forall y.\ y > 0 \ \rightarrow\ rem(x, y) < y$.

Let y be an arbitrary positive integer

Case $x < y$:

$$\begin{array}{rcll} rem(x, y) & = & x & \text{by (R0)} \\ & < & y & \text{case} \end{array}$$

Case $\neg(x < y)$:

Then there is natural number n, $n < x$ s.t. $x = n + y$

$$\begin{array}{rcll} rem(x, y) & = & rem(n + y, y) & x = n + y \\ & = & rem(n, y) & \text{(R1)} \\ & < & y & \text{IH } (x' \mapsto n, y' \mapsto y) \\ & & & \text{since } n < x \text{ and } y > 0 \end{array}$$

Well-founded Induction I

A binary predicate $\prec$ over a set S is a well-founded relation iff there does not exist an infinite decreasing sequence

$$s_1 \succ s_2 \succ s_3 \succ \cdots \text{ where } s_i \in S$$

Note: where $s \prec t$ iff $t \succ s$

Examples:

- $<$ is well-founded over the natural numbers.
  Any sequence of natural numbers decreasing according to $<$ is finite:

  $$1023 > 39 > 30 > 29 > 8 > 3 > 0.$$

- $<$ is not well-founded over the rationals in $[0, 1]$.

  $$1 > \frac{1}{2} > \frac{1}{3} > \frac{1}{4} > \cdots$$

  is an infinite decreasing sequence.

Well-founded Induction II

- $<$ is not well-founded over the integers:

$$7200 > \ldots > 217 > \ldots > 0 > \ldots > -17 > \ldots$$

- The strict sublist relation $\prec_c$ is well-founded over the set of all lists.
- The relation

$$F \prec G \text{ iff } F \text{ is a strict subformula of } G$$

  is well-founded over the set of formulae.

Well-founded Induction Principle

For theory T and well-founded relation $\prec$,
the axiom schema (well-founded induction)

$$(\forall n.\ (\forall n'.\ n' \prec n \ \rightarrow \ F[n']) \ \rightarrow \ F[n]) \ \rightarrow \ \forall x.\ F[x]$$

for Σ-formulae $F[x]$ with one free variable x.

To prove $\forall x.\ F[x]$, i.e.,
$F[x]$ is T-valid for every x,
it suffices to show

- inductive step: For arbitrary n,
  assume inductive hypothesis, i.e.,
  $F[n']$ is T-valid for every n', such that $n' \prec n$
  then prove
  $F[n]$ is T-valid.

Complete induction in T_{PA} is a specific instance of well-founded induction, where the well-founded relation $\prec$ is $<$.

Lexicographic Relation

Given pairs $(S_i, \prec_i)$ of sets S_i and well-founded relations $\prec_i$

$$(S_1, \prec_1), \ldots, (S_m, \prec_m)$$

Construct

$$S = S_1 \times \ldots \times S_m;$$

i.e., the set of m-tuples $(s_1, \ldots, s_m)$ where each $s_i \in S_i$.

Define lexicographic relation $\prec$ over S as

$$\underbrace{(s_1, \ldots, s_m)}_{s} \prec \underbrace{(t_1, \ldots, t_m)}_{t} \Leftrightarrow \bigvee_{i=1}^{m} \left(s_i \prec_i t_i \wedge \bigwedge_{j=1}^{i-1} s_j = t_j \right)$$

for $s_i, t_i \in S_i$.

- If $(S_1, \prec_1), \ldots, (S_m, \prec_m)$ are well-founded, so is $(S, \prec)$.

Example: $S = \{A, \cdots, Z\}$, $m = 3$, $CAT \prec DOG$, $DOG \prec DRY$, $DOG \prec DOT$.

Example: For the set $\mathbb{N}^3$ of triples of natural numbers with the lexicographic relation $\prec$,

$$(5, 2, 17) \prec (5, 4, 3)$$

Lexicographic well-founded induction principle

For theory T and well-founded lexicographic relation $\prec$,

$$(\forall \bar{n}.\ (\forall \bar{n}'.\ \bar{n}' \prec \bar{n} \ \rightarrow\ F[\bar{n}']) \ \rightarrow\ F[\bar{n}]) \ \rightarrow\ \forall \bar{x}.\ F[\bar{x}]$$

for Σ_T-formula $F[\bar{x}]$ with free variables $\bar{x}$, is T-valid.

Same as regular well-founded induction, just

$$n \ \Rightarrow\ \text{tuple } \bar{n} = (n_1, \ldots, n_m) \qquad x \ \Rightarrow\ \text{tuple } \bar{x} = (x_1, \ldots, x_m)$$
$$n' \ \Rightarrow\ \text{tuple } \bar{n}' = (n'_1, \ldots, n'_m)$$

Example: Puzzle

Bag of red, yellow, and blue chips
If one chip remains in the bag – remove it (empty bag – the process terminates)
Otherwise, remove two chips at random:

1. If one of the two is red –
   don't put any chips in the bag
2. If both are yellow –
   put one yellow and five blue chips
3. If one of the two is blue and the other not red –
   put ten red chips

Does this process terminate?

Proof: Consider

- Set $S : \mathbb{N}^3$ of triples of natural numbers and

- Well-founded lexicographic relation $<_3$ for such triples, e.g.

$$(11, 13, 3) \not<_3 (11, 9, 104) \qquad (11, 9, 104) <_3 (11, 13, 3)$$

Let y, b, r be the yellow, blue, and red chips in the bag before a move.
Let y', b', r' be the yellow, blue, and red chips in the bag after a move.

Show

$$(y', b', r') <_3 (y, b, r)$$

for each possible case. Since $<_3$ well-founded relation
$\Rightarrow$ only finite decreasing sequences $\Rightarrow$ process must terminate

1. If one of the two removed chips is red –
   do not put any chips in the bag

$$\left.\begin{array}{c}(y-1,b,r-1)\\(y,b-1,r-1)\\(y,b,r-2)\end{array}\right\} <_3 (y,b,r)$$

2. If both are yellow –
   put one yellow and five blue

$$(y-1,b+5,r) <_3 (y,b,r)$$

3. If one is blue and the other not red –
   put ten red

$$\left.\begin{array}{c}(y-1,b-1,r+10)\\(y,b-2,r+10)\end{array}\right\} <_3 (y,b,r)$$

Example: Ackermann function

Theory $T_{\mathbb{N}}^{ack}$ is the theory of Presburger arithmetic $T_{\mathbb{N}}$ (for natural numbers) augmented with

Ackermann axioms:

- $\forall y.\ ack(0, y) = y + 1$ (L0)
- $\forall x.\ ack(x + 1, 0) = ack(x, 1)$ (R0)
- $\forall x, y.\ ack(x + 1, y + 1) = ack(x, ack(x + 1, y))$ (S)

Ackermann function grows quickly:

$ack(0, 0) = 1$
$ack(1, 1) = 3$
$ack(2, 2) = 7$
$ack(3, 3) = 61$

$$ack(4, 4) = 2^{2^{2^{2^{16}}}} - 3$$

Proof of termination

Let $<_2$ be the lexicographic extension of $<$ to pairs of natural numbers.

(L0) $\forall y.\ ack(0, y) = y + 1$
does not involve recursive call

(R0) $\forall x.\ ack(x + 1, 0) = ack(x, 1)$
$(x + 1, 0) >_2 (x, 1)$

(S) $\forall x, y.\ ack(x + 1, y + 1) = ack(x, ack(x + 1, y))$
$(x + 1, y + 1) >_2 (x + 1, y)$
$(x + 1, y + 1) >_2 (x, ack(x + 1, y))$

No infinite recursive calls $\Rightarrow$ the recursive computation of $ack(x, y)$ terminates for all pairs of natural numbers.

Proof of property

Use well-founded induction over $<_2$ to prove

$$\forall x, y.\ ack(x, y) > y$$

is $T_{\mathbb{N}}^{ack}$ valid.

Consider arbitrary natural numbers x, y.
Assume the inductive hypothesis

$$\forall x', y'.\ \overline{(x', y') <_2 (x, y)} \rightarrow \underbrace{ack(x', y') > y'}_{F[x', y']} \qquad \text{(IH)}$$

Show

$$F[x, y] : ack(x, y) > y.$$

Case $x = 0$:

$$ack(0, y) = y + 1 > y \qquad \text{by (L0)}$$

Case $x > 0 \land y = 0$:

$$ack(x, 0) = ack(x-1, 1) \qquad \text{by (R0)}$$

Since

$$(\underbrace{x-1}_{x'}, \underbrace{1}_{y'}) <_2 (x, y)$$

Then

$$ack(x-1, 1) > 1 \qquad \text{by (IH) } (x' \mapsto x-1, y' \mapsto 1)$$

Thus

$$ack(x, 0) = ack(x-1, 1) > 1 > 0$$

Case $x > 0 \land y > 0$:

$$ack(x,y) = ack(x-1, ack(x, y-1)) \qquad \text{by (S)} \tag{1}$$

Since

$$(\underbrace{x-1}_{x'}, \underbrace{ack(x,y-1)}_{y'}) <_2 (x,y)$$

Then

$$ack(x-1, ack(x,y-1)) > ack(x, y-1) \tag{2}$$

by (IH) $(x' \mapsto x-1, y' \mapsto ack(x, y-1))$.

Furthermore, since

$$(\underbrace{x}_{x'}, \underbrace{y-1}_{y'}) <_2 (x, y)$$

then

$$ack(x, y-1) > y - 1 \tag{3}$$

By (1)–(3), we have

$$ack(x,y) \overset{(1)}{=} ack(x-1, ack(x, y-1)) \overset{(2)}{>} ack(x, y-1) \overset{(3)}{>} y - 1$$

Hence

$$ack(x,y) > (y-1) + 1 = y$$

Structural Induction

How do we prove properties about logical formulae themselves?

Structural induction principle

To prove a desired property of formulae,

> inductive step: Assume the inductive hypothesis, that for arbitrary formula F, the desired property holds for every strict subformula G of F.
> Then prove that F has the property.

Since atoms do not have strict subformulae, they are treated as base cases.

Note: "strict subformula relation" is well-founded

Example: Prove that

Every propositional formula F is equivalent to a propositional formula F' constructed with only $\top$, $\vee$, $\neg$ (and propositional variables)

Base cases:

$$F : \top \;\Rightarrow\; F' : \top$$

$$F : \bot \;\Rightarrow\; F' : \neg\top$$

$$F : P \;\Rightarrow\; F' : P \quad \text{for propositional variable } P$$

Inductive step:
Assume as the inductive hypothesis that G, G_1, G_2 are equivalent to G', G_1', G_2' constructed only from $\top$, $\vee$, $\neg$ (and propositional variables).

$$
\begin{array}{lcl}
F : \neg G & \Rightarrow & F' : \neg G' \\
F : G_1 \vee G_2 & \Rightarrow & F' : G_1' \vee G_2' \\
F : G_1 \wedge G_2 & \Rightarrow & F' : \neg(\neg G_1' \vee \neg G_2') \\
F : G_1 \rightarrow G_2 & \Rightarrow & F' : \neg G_1' \vee G_2' \\
F : G_1 \leftrightarrow G_2 & \Rightarrow & (G_1' \rightarrow G_2') \wedge (G_2' \rightarrow G_1') \Rightarrow F' : \ldots
\end{array}
$$

Each F' is equivalent to F and is constructed only by $\top$, $\vee$, $\neg$ by the inductive hypothesis.